Intrinsic aspects of Hamilton-Jacobi separability

Goedele Waeyaert

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Promotoren: Prof. Dr. F. Cantrijn, Prof. Dr. W. Sarlet

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INTRODUCTION

Hamilton-Jacobi theory is a topic in mathematical physics, and in classical mechanics and classical field theory in particular, that has always attracted a lot of attention. Different aspects of it, as well as several generalizations of the original setting, have been studied intensively. From the very long list of books and papers related to the subject, we just mention some recent contributions: [3], [10], [13], [22], [23], [44], [49] and [60]. In this thesis, we will focus in particular on aspects related to separability of the Hamilton-Jacobi equation, see for instance [7], [9] and [11].

The principal aim of this dissertation can be formulated in one sentence as follows: we intend to give an intrinsic formulation of some results in the context of Hamiltonian systems that are directly or indirectly related to time-dependent systems for which the Hamilton-Jacobi equation can be solved by the method of separation of variables.

The Hamilton-Jacobi equation for a Hamiltonian $H(t, q, p)$ is a first-order partial differential equation for a function $S(t, q)$ given by

$$H\left(t, q, \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0.$$

A complete integral $S(t, q, \alpha)$ is a solution of the Hamilton-Jacobi equation depending on n independent constants $\alpha_1, \ldots, \alpha_n$ and satisfying

$$\det\left(\frac{\partial^2 S}{\partial q^i\,\partial \alpha_j}\right) \neq 0.$$

In general the Hamilton-Jacobi equation is very hard to solve but in certain cases one can do it by the method of separation of variables. This means that one tries to find a complete solution of the form

$$S(t, q, \alpha) = S_0(t, \alpha) + \sum_{i=1}^{n} S_i(q^i, \alpha).$$

If such a solution exists, the Hamiltonian is called separable.

Since the middle of the nineteenth century the method of separation of variables has been studied extensively. A lot of interesting results for autonomous, i.e. time-independent systems, are known. The first coordinate dependent conditions for separability, like the ones of Liouville and Stäckel, are specifically for Hamiltonians separable in orthogonal coordinates. In 1904 Levi-Civita [37] found a test for the separability of a given Hamiltonian system. He stated that a general Hamiltonian $H(q, p)$ is separable in the coordinates (q^i, p_i) if and only if

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial^2 H}{\partial q^i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial q^i\,\partial p_j}\frac{\partial H}{\partial q^j}\right) = \frac{\partial H}{\partial q^i}\left(\frac{\partial^2 H}{\partial p_i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial p_i\,\partial p_j}\frac{\partial H}{\partial q^j}\right),$$

where there is no summation over repeated indices and $i, j = 1, \ldots, n$, $i \neq j$. The Levi-Civita conditions are obviously interesting: they provide a straightforward test for separability. However, they suffer from a major disadvantage. They can merely test whether the Hamilton-Jacobi equation is separable in the given coordinates: one has to be lucky to have chosen separation coordinates already for the test to give a positive result. Moreover they don't tell you if (other) separation coordinates exist and how you can construct them. This is exactly the advantage of intrinsic, i.e. coordinate independent results, such as the results of Eisenhart and Benenti.

For time-dependent systems much less results are known. One important result is the generalization of the Levi-Civita conditions which was done by Forbat in 1944 [30]. He showed that a time-dependent Hamiltonian $H(t, q, p)$ is separable in the coordinates (t, q^i, p_i) if and only if

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial^2 H}{\partial q^i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial q^i\,\partial p_j}\frac{\partial H}{\partial q^j}\right) = \frac{\partial H}{\partial q^i}\left(\frac{\partial^2 H}{\partial p_i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial p_i\,\partial p_j}\frac{\partial H}{\partial q^j}\right),$$

$$\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial q^i\,\partial t} = \frac{\partial H}{\partial q^i}\frac{\partial^2 H}{\partial p_i\,\partial t},$$

where there is again no summation over repeated indices and $i, j = 1, \ldots, n$, $i \neq j$. Unfortunately they have the same weak point as the Levi-Civita conditions: again one has to be lucky to be working already in separation coordinates. But in contrast to the results for autonomous Hamiltonian systems, there exists no coordinate independent test for separability of a time-dependent Hamiltonian. Therefore, the primary aim of the first part of our research is to develop an intrinsic formulation of Forbat's conditions. We want to obtain a test for the existence of separation coordinates which in principle can be carried out in any given coordinate chart and should then provide information about the way separation coordinates can be constructed. Note that, as in the classical approach to the Hamilton-Jacobi equation, we only consider separation coordinates which can be determined from the original coordinates by a point transformation.

There is a variety of possible differential geometric models for time-dependent Hamiltonian systems, but in many papers a trivial bundle $\mathbb{R} \times M$ is chosen as model for the extended configuration space from the outset. This is fine as long as one keeps in mind that for applications of our research, it is essential to allow for time-dependent coordinate transformations, i.e. transformations of the form $(t = t, Q^i = Q^i(t, q))$ where (t, q^i) denote the coordinates on $\mathbb{R} \times M$, and such coordinate transformations do not respect the product structure! Therefore, a more convenient approach is the one which starts from a bundle over $\mathbb{R}$, $\tau : E \to \mathbb{R}$ with $\dim E = n + 1$ and local coordinates on E denoted by (t, q^i). For the analysis of intrinsic aspects of time-dependent Lagrangian systems a convenient setting is then the first jet bundle $J^1\tau$, which is an affine bundle over E with induced bundle coordinates $(t, q^i, \dot{q}^i)$. For the study of the time-dependent Hamiltonian framework the dual bundle of $J^1\tau$ is the more appropriate setting. This is the quotient bundle $T^*E/\langle dt\rangle$ which is denoted by $J^1\tau^*$.

In the study and characterization of Hamilton-Jacobi separability Poisson-Nijenhuis manifolds are often involved. A Poisson manifold, with associated Poisson map P, and a (1,1) tensor field R determine a Poissson-Nijenhuis manifold if PR defines a second Poisson tensor compatible with the original one. Hence, a Poisson-Nijenhuis manifold is an example of a bi-Hamiltonian manifold. The tensor field R is then called the recursion operator. An interesting result, in the case that the recursion operator has n distinct eigenvalues at every point, is that there exist so-called Darboux-Nijenhuis coordinates. These are in fact Darboux coordinates for the Poisson tensor in terms of which the recursion operator is moreover in diagonal form.

A well-known example of a Poisson-Nijenhuis structure on the cotangent bundle is determined by a (1,1) tensor field J on its base manifold. If the Nijenhuis torsion of J vanishes then the complete lift $\tilde{J}$ of J to the cotangent bundle also has vanishing Nijenhuis torsion. Then $\tilde{J}$, together with the inverse of the canonical symplectic structure ω, defines a Poisson-Nijenhuis structure on the cotangent bundle. When it comes to an intrinsic characterization of standard Hamilton-Jacobi separability for autonomous Hamiltonian systems, it is exactly this construction of a Poisson-Nijenhuis structure which plays an important role.

This served as a source of inspiration for our intrinsic formulation of Forbat's conditions. As said before, to describe time-dependent Hamiltonian systems the appropriate setting is $J^1\tau^*$. This means that we first need to extend the theory of lifting geometric objects from the cotangent bundle T^*E to $J^1\tau^*$. One of the main objectives in this first part of our research is to come to an intrinsic definition of the complete lift of a type (1,1) tensor field from E to $J^1\tau^*$ and to understand all features of its construction. This is in particular useful in view of the construction of a Poisson-Nijenhuis structure on $J^1\tau^*$.

These results make it possible to give an intrinsic formulation of Forbat's conditions in an elegant way. Now, a Hamiltonian is a section h of the line bundle $\rho : T^*E \to J^1\tau^*$. Locally, h defines a function H on $J^1\tau^*$, determined by $h : (t, q, p) \mapsto (t, q, p_0 = -H(t, q, p), p)$. The associated Hamiltonian vector field on $J^1\tau^*$ is locally of the form

$$X_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

One may hope to develop an intrinsic model for the separability issue directly on $J^1\tau^*$, the manifold where X_h lives. However, one has to be cautious: it is well known that the Hamiltonian function H on $J^1\tau^*$ picks up extra terms under a time-dependent canonical transformation which come from the induced transformation of p_0 on T^*E. So in a way, T^*E has to remain in the picture. Our aim is to explore in detail how objects on T^*E relate to objects on $J^1\tau^*$ and vice versa. For short, we can say that it is an interplay between the complete lifts of a (1,1) tensor field on E to both $J^1\tau^*$ and T^*E that leads to the identification of related distributions on both manifolds. The integrability of these distributions, which is a coordinate free condition, is shown to produce exactly Forbat's conditions for separability of the time-dependent Hamilton-Jacobi equation in appropriate coordinates.

In the second part of this dissertation we consider a special class of systems of partially decoupled second-order differential equations, so-called driven cofactor systems. They were introduced and studied in detail in Euclidean space [41]. Later on they were partially generalized to Riemannian manifolds [54]. It is our intention to complete this generalization.

In the study of driven cofactor systems on a Riemannian manifold special conformal Killing tensors play a prominent role. With respect to the Riemannian metric g they must satisfy equations of the form

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$$

where $|k$ denotes the covariant derivative and $\alpha_i = \partial(\operatorname{tr} J)/\partial q^i$. They are in fact conformal Killing tensors of a specific form. Special conformal Killing tensors have a lot of interesting properties. The cofactor tensor of a nonsingular special conformal Killing tensor is for example always a Killing tensor. Additionally it can be calculated that the Nijenhuis torsion of a special conformal Killing tensor automatically vanishes. Also in other areas of research, special conformal Killing tensors play a prominent role, for example in the results of Benenti in the context of separation of variables for the Hamilton-Jacobi equation.

On a Riemannian manifold driven cofactor systems are differential equations of the form

$$\begin{aligned}
\ddot{y}^i &= -\Gamma^i_{jk}(y)\dot{y}^j\dot{y}^k + Q^i(y), && i = 1,\ldots,m\\
\ddot{x}^a &= -\Gamma^a_{bc}(x)\dot{x}^b\dot{x}^c + Q^a(y,x) && a = 1,\ldots,n, \quad (\text{here } n+m = \dim M).
\end{aligned}$$

They are clearly partially decoupled: the y-system is referred to as the driving system and the remaining x-equations as the driven system. In addition, it is assumed that the overall system is of cofactor type and that the force terms Q^a come from a potential, parametrically depending on the driving coordinates, such that the driven system has a standard Hamiltonian representation.

For driven cofactor systems in Euclidean space, it was shown that there exists a time-dependent canonical transformation which has the effect of reducing the Hamilton-Jacobi problem of the (a priori time-dependent) driven part of the system into that of an equivalent autonomous system of Stäckel type. The generalization of this result is not obvious, as it seems to be impossible to obtain this result by only making use of a time-dependent point transformation. Consequently, our theory related to Forbat's conditions is not directly applicable to this case. Therefore, we focus in our research on the nature of this remarkable canonical transformation and so complete the study of driven cofactor systems on Riemannian manifolds.

Outline of the dissertation

In the first chapter we sketch the mathematical background of this thesis. First we describe Poisson manifolds. Subsequently we study derivations: we recall some definitions and theorems from the theory of Frölicher and Nijenhuis about derivations of vector-valued differential forms. Beside this we also give a short summary of the calculus along a tangent bundle projection $\tau : TM \to M$ and we discuss derivation operators along τ. The three main topics in this introductionary chapter are, however, the central background topics for this dissertation, namely: Poisson-Nijenhuis manifolds, special conformal Killing tensors and the separability of the Hamilton-Jacobi equation.

We start Chapter 2 with the introduction of the dual $J^1\tau^*$ of the first jet bundle $J^1\tau$ of a fibre bundle $\tau : E \to \mathbb{R}$. We then first recall the main lifting operations to a cotangent bundle, since they serve as a source of inspiration for our research. The new contributions begin in Section 2.4 with the description of various lifts of vector fields and 1-forms on E to $J^1\tau^*$. Ways of lifting type (1,1) tensor fields from E to $J^1\tau^*$ are introduced in Section 2.5. Further properties relating the constructions of the two preceding sections are derived in Section 2.5.1. They are indispensable for proving, in Section 2.6, that the canonical Poisson structure on $J^1\tau^*$, together with the complete lift of a type (1,1) tensor field R on E with vanishing Nijenhuis torsion, determines a Poisson-Nijenhuis structure on $J^1\tau^*$. The construction of Darboux-Nijenhuis coordinates for this structure is explained in detail in Section 2.6.1.

Making use of the results in Chapter 2, we establish in Chapter 3 an intrinsic version of Forbat's conditions for separability. We start from a type (1,1) tensor field R on $J^1\tau^*$ with the property $R(dt) = 0$. Under an assumption of diagonalizability of

R, the complete lift on T^*E gives rise to an interesting distribution associated to any function F on T^*E, and we characterize its integrability in Section 3.2. In the case of a function H defining the image of a section $h : J^1\tau^* \to T^*E$, there is a corresponding distribution on $J^1\tau^*$. The interplay between the two distributions is studied in detail in Section 3.3. The integrability of both distributions is claimed to be an intrinsic version of Forbat's conditions for separability and we prove this claim in Section 3.4 by showing that we indeed recover Forbat's conditions in Darboux-Nijenhuis coordinates for the Poisson-Nijenhuis structures under consideration. We end the chapter with some illustrating examples in Section 3.5.

Chapter 4 is devoted to the study of driven cofactor systems. We start with recalling the definition of a cofactor system in Euclidean space as well as on a Riemannian manifold. In Section 4.2 we first recall the intrinsic characterization of submersive systems to consider subsequently the intrinsic definition of a driven cofactor system. In Section 4.3, we develop the algorithm which leads to the identification of $n+1$ quadratic first integrals, where n (the dimension of the driven system) of them are integrals of the driven system along solutions of the driving one. A key issue for understanding the nature of the driven system is the identification of a special conformal Killing tensor for its proper metric. In Section 4.5, we start by identifying Darboux coordinates for the symplectic form associated to the special conformal Killing tensor of the complete system: they are obtained by suitably modifying the momenta. We then gradually develop arguments to come to an even better selection of modified momenta, which takes the specific decoupling properties of our system into account and are shown to be related to a time-dependent (standard) canonical transformation for the driven part of the system. In Section 4.6, we prove that the application of this canonical transformation, followed by one which comes from using eigenfunctions as new coordinates, produces the rather miraculous effect of reducing the driven system essentially to an autonomous Stäckel type system. The proofs in Section 4.6 are partly based on simple, indirect arguments, but they are supported also by explicit computations about the structure of all first integrals, which are presented in Section 4.7. A couple of illustrative examples are presented in Section 4.8.

References. Most of the work presented here has already been published. The new contributions in Chapter 2 begin in Section 2.4 and have appeared in [56]. The intrinsic characterization of Forbat's conditions, which is the subject of Chapter 3, was published in [61]. The new results in the last chapter, which begin in Section

4.3, are based on [55].

CHAPTER

1 PRELIMINARIES

In this first introductory chapter we collect some standard results concerning derivations, Poisson-Nijenhuis manifolds, special conformal Killing tensors and separable systems. This way we hope to present the general framework in which our research fits and to provide a concise reference for later use. We also recall some definitions, mainly to fix notations and terminology. We will skip most of the technical aspects, for which reference will be made to the literature.

1.1 Poisson manifolds

Let M be a manifold of dimension m. Then $C^\infty(M)$ denotes the set of smooth functions on M, $\mathcal{X}(M)$ and $\mathcal{X}^*(M)$ are, respectively, the $C^\infty(M)$-module of vector fields and 1-forms on M. The basic reference for this section is the book by Libermann and Marle [38].

Definition 1.1. *A Poisson structure on a manifold M is a bilinear map*

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M) : (f, g) \mapsto \{f, g\},$$

which satisfies the following properties

1. *it is skew symmetric:* $\{f, g\} = -\{g, f\}$,
2. *it satisfies the Jacobi identity:* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$,
3. *it obeys the Leibniz rule:* $\{f, gh\} = g\{f, h\} + \{f, g\}h$,

for all $f, g, h \in C^\infty(M)$.

We call $\{.,.\}$ the *Poisson bracket* and a manifold equipped with a Poisson structure is called a *Poisson manifold*.

On every Poisson manifold M there exists a unique bivector field Λ such that for every pair (f, g) of functions on M

$$\{f, g\} = \Lambda(df, dg).$$

The bivector field Λ is called the *Poisson tensor* or *Poisson bivector* associated to the Poisson bracket $\{.,.\}$. The Jacobi identity is equivalent with the vanishing of the Schouten bracket of Λ with itself: $[\Lambda, \Lambda] = 0$. The associated *Poisson map* $P : \mathcal{X}^*(M) \to \mathcal{X}(M)$ is defined by

$$\Lambda(\alpha, \beta) = \langle P(\alpha), \beta\rangle, \qquad \forall \alpha, \beta \in \mathcal{X}^*(M).$$

The *Hamiltonian vector field* associated with a function $f \in C^\infty(M)$ is the unique vector field X_f on M such that

$$X_f(g) = -\{f, g\} = -P(df)(g), \qquad \forall g \in C^\infty(M).$$

In case of a symplectic manifold (M, ω), we can define a nondegenerate Poisson structure by

$$\{f, g\} = \omega(X_f, X_g),$$

where $i_{X_f}\omega = -df$.

Definition 1.2. *Let* $(M_1, \{.,.\}_1)$ *and* $(M_2, \{.,.\}_2)$ *be two Poisson manifolds and* $\phi : M_1 \to M_2$ *a differentiable map. Then* ϕ *is a Poisson morphism if*

$$\phi^*\{f, g\}_2 = \{\phi^* f, \phi^* g\}_1, \qquad \forall f, g \in C^\infty(M_2).$$

1.2 Derivations

In this section we describe derivations. First we define derivations on the algebra of differential forms on a manifold M, based on the theory of Frölicher and Nijenhuis. Secondly, we study an extension in the context of the calculus along the tangent bundle projection $\tau : TM \to M$.

1.2.1 The Frölicher-Nijenhuis theory

We recall briefly some definitions and theorems from the theory of Frölicher and Nijenhuis about derivations of vector-valued differential forms. For more details and proofs we refer to [29] and [34].

Definition 1.3. *A derivation D of degree r on $\Lambda(M)$, the algebra of differential forms on M, is a map $D : \Lambda(M) \to \Lambda(M)$, which satisfies*

1. $D(\Lambda^p(M)) \subset \Lambda^{p+r}(M)$,
2. $D(\alpha + a\beta) = D(\alpha) + aD(\beta), \qquad \alpha, \beta \in \Lambda^p(M), a \in \mathbb{R}$,
3. $D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + (-1)^{pr}\alpha \wedge D(\beta), \qquad \alpha \in \Lambda^p(M), \beta \in \Lambda^q(M)$.

Lemma 1.4. *A derivation D is a local operator, i.e. if $\omega, \theta \in \Lambda^p(M)$ have the property that $\omega|_U = \theta|_U$, where U is an open subset of M, then $(D\omega)|_U = (D\theta)|_U$.*

Corollary 1.5. *Every derivation is determined by its action on $\Lambda^0(M) = C^\infty(M)$ and $\Lambda^1(M)$.*

It immediately follows that any derivation of degree smaller than or equal to -2 is trivial.

Proposition 1.6. *The commutator of two derivations D_1 of degree r_1 and D_2 of degree r_2,*

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1,$$

is a derivation of degree $r_1 + r_2$. The set of derivations on $\Lambda(M)$ is a graded Lie algebra: the commutator

1. *is graded skew symmetric,*

$$[D_1, D_2] = -(-1)^{r_1 r_2}[D_2, D_1],$$

2. *satisfies the graded Jacobi identity,*

$$(-1)^{r_1 r_3}[D_1, [D_2, D_3]] + (-1)^{r_1 r_2}[D_2, [D_3, D_1]] + (-1)^{r_3 r_2}[D_3, [D_1, D_2]] = 0,$$

where D_1, D_2, D_3 are derivations of degree r_1, r_2, r_3 respectively.

Two special types of derivations can be defined.

Definition 1.7. *A derivation D on $\Lambda(M)$ is said to be of type i_* if it acts trivially on $C^\infty(M)$.*

Definition 1.8. *Let L be a vector-valued form of degree $l+1$. Define $i_L : \Lambda^k(M) \to \Lambda^{k+l}(M)$, a derivation of degree l: for $\omega \in \Lambda^k(M)$ and for $X_1, \ldots, X_{l+k} \in \mathcal{X}(M)$,*

$$(i_L\omega)(X_1, \ldots, X_{l+k}) =$$
$$\frac{1}{(k-1)!(l+1)!} \sum_{\sigma \in S_{l+k}} (sgn\ \sigma)\omega(L(X_{\sigma(1)}, \ldots, X_{\sigma(l+1)}), X_{\sigma(l+2)}\ \ldots, X_{\sigma(l+k)})$$

and for $\omega \in C^\infty(M)$ $(k=0)$,

$$i_L\omega = 0.$$

Hereby S_{l+k} is the group of all permutations of $(1, \ldots, k+l)$ and $sgn\,\sigma$ is the sign of the permutation. It is clear that i_L is a derivation of type i_*. The converse is also true.

Theorem 1.9. *For every derivation D of degree r $(r \geq -1)$ of type i_*, there exists a vector-valued form L of degree $r+1$ such that $D = i_L$.*

An example of a derivation of type i_* (of degree -1) is the contraction of a vector field with a k-form ω: $i_X\omega$.

The second type of derivations is defined as follows.

Definition 1.10. *A derivation D on $\Lambda(M)$ is said to be of type d_* if it commutes with the exterior derivative d: $[D, d] = 0$.*

Note that a derivation of type d_* is completely determined by its action on $C^\infty(M)$.

Definition 1.11. *Let L be a vector-valued form of degree l. Define a derivation d_L, of degree l, by*

$$d_L = [i_L, d].$$

It immediately follows that d_L is a derivation of type d_*. Again, also the converse is true.

Theorem 1.12. *For every derivation D of degree r $(r > -1)$ of type d_*, there exists a vector-valued form L of degree r such that $D = d_L$.*

Well known examples of derivations of type d_* are the exterior derivative d and the Lie derivative $\mathcal{L}_X = [i_X, d]$.

One of the main results of this section is the following.

Theorem 1.13. *For every derivation D of degree r there exist vector-valued forms L_1 and L_2 of degree $r+1$ and r respectively, such that*

$$D = i_{L_1} + d_{L_2}.$$

Moreover, this decomposition is unique.

The commutator of two derivations of type d_* is also of type d_* and it can be used to define the *Nijenhuis bracket* of two vector-valued forms.

Proposition 1.14. *Given two vector-valued forms K and L of degree k and l respectively, there exists a uniquely determined vector-valued form $[K, L]$ of degree $k+l$, defined by*

$$[d_K, d_L] = d_{[K,L]},$$

$[K, L]$ is called the Nijenhuis bracket of K and L.

For $k = 0$, i.e. K is a vector field, $[K, L]$ is the Lie derivative of L with respect to K. If also $l = 0$, $[K, L]$ is the usual Lie bracket of vector fields.

Definition 1.15. *The Nijenhuis torsion of a* $(1,1)$ *tensor field* R*, is a type* $(1,2)$ *tensor field defined by*

$$N_R(X,Y) = [R(X), R(Y)] + R^2([X,Y]) - R([R(X),Y]) - R([X,R(Y)]) \tag{1.1}$$

with $X, Y \in \mathcal{X}(M)$.

For a $(1,1)$ tensor field R (a vector-valued form of degree 1), there is a direct link between the Nijenhuis torsion and the Nijenhuis bracket, namely

$$[R, R] = 2N_R.$$

It follows that $d_R^2 = 0$ iff the Nijenhuis torsion of R is zero:

$$2d_R^2 = [d_R, d_R] = d_{[R,R]} = d_{2N_R}.$$

Another interesting property is the following.

Lemma 1.16. *Let* L *be a type* $(1,1)$ *tensor field on an arbitrary manifold* M*. Then, for any 1-form* α *and vector fields* X, Y *on* M*:*

$$d_L(L\alpha)(X,Y) = d\alpha(LX, LY) + \alpha\big(N_L(X,Y)\big)^1. \tag{1.2}$$

Proof. Since $d_L = i_L d - d i_L$, we have

$$d_L(L\alpha)(X,Y) = d(L\alpha)(LX,Y) + d(L\alpha)(X,LY) - d(L^2\alpha)(X,Y).$$

Using the general property $d\alpha(X,Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X,Y])$, this easily reduces to

$$\begin{aligned} d_L(L\alpha)(X,Y) &= \mathcal{L}_{LX}((L\alpha)(Y)) - \mathcal{L}_{LY}((L\alpha)(X)) \\ &\quad - (L\alpha)\big([LX,Y]\big) - (L\alpha)\big([X,LY]\big) + (L^2\alpha)\big([X,Y]\big), \\ &= \mathcal{L}_{LX}(\alpha(LY)) - \mathcal{L}_{LY}(\alpha(LX)) - \alpha\big([LX,LY]\big) + \alpha\big(N_L(X,Y)\big), \end{aligned}$$

from which the result now follows. □

[1] Throughout this thesis we make no notational distinction between the action of a (1,1) tensor field on vector fields and its adjoint action on 1-forms; for example, for $X \in \mathcal{X}(M)$ and $\alpha \in \mathcal{X}^*(M)$, we have $\langle L(X), \alpha\rangle = \langle X, L(\alpha)\rangle$. Moreover, to avoid an overload of notations, we will often write $L\alpha$ or LX instead of $L(\alpha)$ or $L(X)$ respectively.

In particular, if L has vanishing Nijenhuis torsion, then

$$d_L(L\alpha)(X,Y) = d\alpha(LX,LY). \tag{1.3}$$

Tensor fields with vanishing Nijenhuis torsion have another interesting property, which will play a role further on.

Lemma 1.17. *For any (1,1) tensor field L on a manifold M such that $N_L = 0$, we have*

$$d_L(\det L) = (\det L)d(\operatorname{tr} L). \tag{1.4}$$

Proof. For the determinant of L we have

$$\det L = \frac{1}{n!}\delta^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_n} L^{j_1}_{i_1} L^{j_2}_{i_2} \cdots L^{j_n}_{i_n},$$

where $\delta^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_n}$ is the generalized Kronecker delta:

$$\delta^{i_1 i_2 \dots i_n}_{j_1 j_2 \dots j_n} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_n} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_n}_{j_1} & \delta^{i_n}_{j_2} & \cdots & \delta^{i_n}_{j_n} \end{vmatrix}.$$

So, in local coordinates q^i on M we have

$$\frac{\partial \det L}{\partial q^k} = \frac{1}{(n-1)!}\delta^{i i_2 \dots i_n}_{j j_2 \dots j_n} \frac{\partial L^j_i}{\partial q^k} L^{j_2}_{i_2} \cdots L^{j_n}_{i_n} = \frac{\partial L^j_i}{\partial q^k} A^i_j$$

where A is the cofactor tensor of L, defined by $AL = (\det L)I$. Making use of partial derivatives, the coordinate expression of the Nijenhuis torsion N_L is given by

$$(N_L)^k_{ij} = L^k_l \left(\frac{\partial L^l_i}{\partial q^j} - \frac{\partial L^l_j}{\partial q^i} \right) + L^l_i \frac{\partial L^k_j}{\partial q^l} - L^l_j \frac{\partial L^k_i}{\partial q^l}. \tag{1.5}$$

Thus

$$
\begin{aligned}
d_L(\det L) &= L(d(\det L)) \\
&= L^k_l \frac{\partial \det L}{\partial q^k} dq^l \\
&= L^k_l \frac{\partial L^j_i}{\partial q^k} A^i_j dq^l \\
&= \left(L^j_k \frac{\partial L^k_i}{\partial q^l} - L^j_k \frac{\partial L^k_l}{\partial q^i} + L^k_i \frac{\partial L^j_l}{\partial q^k} \right) A^i_j dq^l \\
&= (\det L) \frac{\partial L^i_i}{\partial q^l} dq^l \\
&= (\det L) d(\operatorname{tr} L).
\end{aligned}
$$

□

1.2.2 Calculus along the tangent bundle projection

We will regularly make use of derivation operators along the tangent bundle projection. Therefore, we give a short summary of the results of papers [45] and [46] where the calculus along the tangent bundle projection $\tau : TM \to M$ is developed.

In general, let $\pi : E \to M$ be a fibre bundle and $\phi : N \to M$ a smooth map. A *section of* E *along* ϕ is a map $\sigma : N \to E$ such that $\pi \circ \sigma = \phi$. If E is a vector bundle, the set of sections along ϕ is a $C^\infty(N)$-module. An interesting case is $E = TM$: a section of E along ϕ is then called a vector field along ϕ. Similarly, if $E = T^*M$ a section of E along ϕ is called a 1-form along ϕ.

In our situation, the special case where ϕ is the tangent bundle projection $\tau : TM \to M$, is particularly interesting. A *vector field along* τ is then a map $X : TM \to TM$ such that for each $v \in T_qM$, $X(v)$ is a tangent vector to M in q. A *1-form along* τ is a map $\alpha : TM \to T^*M$ such that for each $v \in T_qM$, $\alpha(v)$ is a covector at q. Or in other words, the following schemes must be commutative.

$$
\begin{tikzcd}
 & TM \arrow[d, "\tau"] \\
TM \arrow[ur, "X"] \arrow[r, "\tau"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T^*M \arrow[d, "\pi"] \\
TM \arrow[ur, "\alpha"] \arrow[r, "\tau"'] & M
\end{tikzcd}
$$

The vector fields along τ form a $C^\infty(TM)$-module, denoted by $\mathcal{X}(\tau)$. Similarly, $\mathcal{X}^*(\tau)$ is the $C^\infty(TM)$-module of 1-forms along τ. The most elementary example of a vector field X along τ is a *basic vector field*, i.e. a vector field Y on M regarded as an element of $\mathcal{X}(\tau)$: $X = Y \circ \tau$. In natural coordinates (q^i, v^i) on TM the coordinate expression of a general vector field X and general 1-form α along τ is

$$X = X^i(q,v)\frac{\partial}{\partial q^i}, \qquad \alpha = \alpha_i(q,v)dq^i.$$

Consider the following geometrical concepts on a tangent bundle, which we will later on extend to vector fields along τ. An *Ehresmann connection* on $\tau : TM \to M$ is a smooth procedure for defining at each point (q,v) of TM a 'horizontal subspace' of $T_{(q,v)}(TM)$ of the same dimension as T_qM and complementary to the space of vertical vectors $V_{(q,v)}TM = \{w \in T_{(q,v)}(TM) | \tau_* w = 0\}$. Making use of a *second-order differential equation field* (SODE), a specific Ehresmann connection can be defined. A SODE Γ is intrinsically determined by the condition $S(\Gamma) = \Delta$, where S is the vertical endomorphism and Δ the dilation vector field on TM. Then Γ has a coordinate representation of the form

$$\Gamma = v^i\frac{\partial}{\partial q^i} + f^i(q,v)\frac{\partial}{\partial v^i},$$

i.e. it represents the system of second-order differential equations $\ddot{q}^i = f^i(q,v)$. Now, the corresponding connection is determined by the following intrinsic procedure for a horizontal lift of vector fields on M,

$$X \in \mathcal{X}(M) \quad \mapsto \quad X^H \in \mathcal{X}(TM) = \frac{1}{2}(X^C + [X^V, \Gamma]),$$

where X^V and X^C are respectively the (canonically defined) vertical and complete lift of X to TM. These are determined, in coordinates, by the following prescriptions: for $X = X^i(q)\partial/\partial q^i$,

$$X^V = X^i(q)\frac{\partial}{\partial v^i}, \quad X^C = X^i(q)\frac{\partial}{\partial q^i} + v^j\frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial v^i}.$$

In coordinates, we have for the horizontal lift,

$$X^H = X^i\, H_i, \quad \text{with} \quad H_i = \left(\frac{\partial}{\partial q^i}\right)^H = \frac{\partial}{\partial q^i} - \Gamma_i^j\,\frac{\partial}{\partial v^j}\,. \tag{1.6}$$

The functions

$$\Gamma_i^j = -\frac{1}{2}\frac{\partial f^j}{\partial v^i},$$

are the *connection coefficients*.

It is clear that the horizontal and vertical lift still make sense if we allow the components of X to be functions $X^i(q, v)$, meaning that we can naturally extend the domain of both operations to $\mathcal{X}(\tau)$. So, in the presence of a connection, every vector field on $Z \in \mathcal{X}(TM)$ has a unique decomposition into a vertical and horizontal part which are necessarily lifts of vector fields along τ; we can write for example,

$$Z = X^H + Y^V, \quad \text{with} \quad X, Y \in \mathcal{X}(\tau).$$

In coordinates, if Z has the expression

$$Z = Z_1^i\frac{\partial}{\partial q^i} + Z_2^i\frac{\partial}{\partial v^i},$$

X and Y are of the form

$$X = Z_1^i\frac{\partial}{\partial q^i} \qquad \text{and} \qquad Y = (Z_2^i + \Gamma_j^i Z_1^j)\frac{\partial}{\partial q^i}.$$

Now, with the above decomposition in mind, we can define for all $X, Y \in \mathcal{X}(\tau)$, two fundamental self-dual derivations of degree 0, D_X^V en D_X^H:

$$[X^H, Y^V] \;=\; (\mathrm{D}_X^H Y)^V \;-\; (\mathrm{D}_Y^V X)^H. \tag{1.7}$$

D_X^V and D_X^H are called, respectively, the *vertical* and *horizontal covariant derivative*. Note that a derivation D of degree 0 is said to be *self-dual* if $\forall X \in \mathcal{X}(\tau)$, $\alpha \in \mathcal{X}^*(\tau)$,

$$D(\alpha(X)) = D\alpha(X) + \alpha(DX).$$

In coordinates, the vertical and horizontal covariant derivatives D^V_X and D^H_X are determined by the following action on functions $F \in C^\infty(TM)$ and basic vector fields (and then further extended by duality):

$$\mathrm{D}^V_X F = X^i\, V_i(F), \qquad \mathrm{D}^V_X \frac{\partial}{\partial q^i} = 0, \qquad \mathrm{D}^V_X dq^i = 0,$$

$$\mathrm{D}^H_X F = X^i\, H_i(F), \qquad \mathrm{D}^H_X \frac{\partial}{\partial q^i} = X^j V_i(\Gamma^k_j)\frac{\partial}{\partial q^k}, \qquad \mathrm{D}^H_X dq^i = -X^j V_k(\Gamma^i_j) dq^k.$$

Note that for a general basic vector field Y, $\mathrm{D}^V_X Y = 0$.

Furthermore, the decomposition of $\mathcal{L}_\Gamma X^H$ into its horizontal and vertical part identifies the important concepts of the *dynamical covariant derivative* ∇, a self-dual degree 0 derivation on tensor fields along τ, and the *Jacobi endomorphism*, a $(1,1)$ tensor field Φ along τ,

$$\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V.$$

For practical purposes, it suffices to know that, $F \in C^\infty(TM)$,

$$\nabla F = \Gamma(F), \qquad \nabla \frac{\partial}{\partial q^i} = \Gamma^j_i \frac{\partial}{\partial q^j}, \qquad \nabla dq^i = -\Gamma^i_j dq^j, \tag{1.8}$$

$$\Phi^i_j = -\frac{\partial f^i}{\partial q^j} - \Gamma^i_k \Gamma^k_j - \Gamma(\Gamma^i_j)\,.$$

1.3 Poisson-Nijenhuis manifolds

Definition 1.18. *A bi-Hamiltonian manifold is a Poisson manifold M endowed with two compatible Poisson structures $\{.,.\}_0$ and $\{.,.\}_1$. The Poisson structures are compatible if for any $\lambda \in \mathbb{R}$, the linear combination*

$$\{.,.\}_\lambda = \{.,.\}_0 - \lambda\{.,.\}_1$$

also defines a Poisson bracket.

An important example of a bi-Hamiltonian manifold is a Poisson-Nijenhuis manifold.

Definition 1.19. *A Poisson-Nijenhuis manifold [35, 43] is a triple (M, P_0, R) of a Poisson manifold (M, P_0) and a (1,1) tensor field R on M, often called the recursion operator, such that*

1. R commutes with P_0: $P_0 R = R P_0$,

2. R has vanishing Nijenhuis torsion: $N_R = 0$,

3. the Magri-Morosi concomitant [43] μ_{R,P_0} is zero, where

$$\mu_{R,P_0}(\sigma, Z) = \big(\mathcal{L}_{P_0(\sigma)} R\big)(Z) - P_0\big(\mathcal{L}_Z(R(\sigma))\big) + P_0\big(\mathcal{L}_{R(Z)}\sigma\big), \tag{1.9}$$

for all $\sigma \in \mathcal{X}^*(M)$ and $Z \in \mathcal{X}(M)$.

Due to the first condition, the Magri-Morosi concomitant is a tensor field of type $(2,1)$. The last two conditions are the necessary and sufficient conditions for $P_1 = RP_0$ to define a Poisson bracket which is compatible with the one associated with P_0. So Poisson-Nijenhuis manifolds are indeed examples of bi-Hamiltonian manifolds.

We now consider more specifically the case in which P_0 is nondegenerate and therefore comes from a symplectic form ω_0 on M. The first condition in the previous definition implies that the (1,1) tensor field R and ω_0 are such that $\omega_0(R(X), Y) = \omega_0(X, R(Y))$ for every pair of vector fields X and Y on M. Then ω_1, defined by

$$\omega_1 = \omega_0(R(\cdot), \cdot) = \tfrac{1}{2} i_R \omega_0,$$

is a 2-form and the following conditions are equivalent [18]:

$$\mu_{R,P_0} = 0 \qquad \Leftrightarrow \qquad d\omega_1 = 0.$$

If in addition to the above equivalent conditions it is assumed that $N_R = 0$, RP_0 defines a second Poisson structure which is compatible with the original one. The second Poisson bracket is then given by

$$\{f, g\}_1 = \omega_1(X_f, X_g),$$

where X_f is the Hamiltonian vector field corresponding with f with respect to ω_0. Such a Poisson-Nijenhuis manifold where one of the Poisson structures is nondegenerate is sometimes called an *ωN-manifold*.

As an example of the special Poisson-Nijenhuis manifold just described, we consider the case in which M is a cotangent bundle T^*Q, with standard symplectic structure $\omega = d\theta = dp_i \wedge dq^i$. Let J be a type (1,1) tensor field on Q. We define the 2-form ω_1 on T^*Q by

$$\omega_1 = d\theta_1, \qquad \text{with} \qquad \theta_1 = \tau_J^*(\theta) = J^i_j p_i dq^j,$$

with $\tau_J : T^*Q \to T^*Q : (q^i, p_i) \mapsto (q^i, J^i_j p_i)$ the fibre linear map on T^*Q defined by J. Define a tensor field R by $\omega_1 = \omega(R(\cdot), \cdot)$. In [32] it was observed that this tensor field is the complete lift $\tilde{J}$ of J to T^*Q [15],

$$\widetilde{J} = J^i_j \left(\frac{\partial}{\partial q^i} \otimes dq^j + \frac{\partial}{\partial p_j} \otimes dp_i \right) + p_i \left(\frac{\partial J^i_j}{\partial q^k} - \frac{\partial J^i_k}{\partial q^j} \right) \frac{\partial}{\partial p_j} \otimes dq^k.$$

So, we have constructed a tensor field $R = \tilde{J}$ on (T^*Q, ω) which is by definition symmetric with respect to ω, since ω_1 is a 2-form, and for which the corresponding $d\omega_1 = 0$. This implies that the Magri-Morosi concomitant will vanish. Moreover, it can be shown [15] that if $N_J = 0$ this implies $N_{\tilde{J}} = 0$. We conclude that any type (1,1) tensor field on Q with vanishing Nijenhuis torsion defines a Poisson-Nijenhuis structure on T^*Q.

If the recursion operator has n distinct eigenvalues λ_i at every point, the Poisson-Nijenhuis manifold is called *semisimple* and then there exists a special class of coordinates, so-called Darboux-Nijenhuis coordinates.

Definition 1.20. *A set of local coordinates (Q, P) on a $2n$-dimensional Poisson-Nijenhuis manifold (M, P_0, R) is a called a set of Darboux-Nijenhuis coordinates if they satisfy the following conditions*

1. *they are canonical with respect to P_0: $\{Q^i, P_j\}_0 = \delta^i_j$, $\{Q^i, Q^j\}_0 = \{P_i, P_j\}_0 = 0$,*

2. *they put the recursion operator R in diagonal form:*

$$R = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial Q^i} \otimes dQ^i + \frac{\partial}{\partial P_i} \otimes dP_i \right).$$

A tensor field R on M has functionally independent eigenfunctions if there are n functions λ_i such that at each point $m \in M$, $\lambda_i(m)$ is an eigenvalue of $R(m)$ for each i and if the Jacobian matrix $(\partial \lambda_i / \partial q^j)$ is everywhere nonsingular. In this case, it is possible to find n functions μ_i which, along with the eigenvalues λ_i, are Darboux-Nijenhuis coordinates. Such coordinates are called *special Darboux-Nijenhuis coordinates*. A few general references in this respect are [42] and [28]. However, it is hard to find a detailed explanation of the way Darboux-Nijenhuis coordinates arise.

We will now (briefly) discuss the relationship between a Poisson-Nijenhuis manifold and a bi-differential calculus. Namely, a symplectic manifold (M, ω) can be endowed with a Poisson-Nijenhuis structure by a bi-differential calculus ([18] and [24]).

Definition 1.21. *A (simple) bi-differential calculus on $\Lambda(M)$ is a pair (d_1, d_2) of derivations of degree 1 on $\Lambda(M)$, which both have the co-boundary property $d_i^2 = 0$ and commute: $[d_1, d_2] = d_1 d_2 + d_2 d_1 = 0$.*

In particular, let d_1 be the exterior derivative d. According to the Frölicher-Nijenhuis theory, a derivation of degree 1 which (anti-)commutes with d must be of the form $d_2 = d_R$ for some type $(1,1)$ tensor field R on M. The necessary and sufficient condition for d_R to satisfy $d_R^2 = 0$ is that the Nijenhuis torsion N_R of R must be zero.

Theorem 1.22. *Suppose that (d, d_R) is a bi-differential calculus on a symplectic manifold (M, ω) and that R is symmetric with respect to ω. Then (P_0, R), with P_0 the Poisson map corresponding with ω, defines a Poisson-Nijenhuis structure on M.*

This follows easily from the construction of an ωN-manifold.

The benefit of a bi-differential calculus (d, d_R) is that if a function $\chi^{(0)} \in C^\infty(M)$ satisfies $dd_R\chi^{(0)} = -d_R d\chi^{(0)} = 0$, we can inductively define a sequence of functions $\chi^{(l)}$ by the rule

$$d\chi^{(l+1)} = d_R\chi^{(l)}.$$

Now, this sequence has interesting properties in the case in which M is a symplectic manifold.

Theorem 1.23. *[18] Suppose that (d, d_R) is a bi-differential calculus on a symplectic manifold (M, ω) and that R is symmetric with respect to ω. If $dd_R\chi^{(0)} = 0$, then the functions $\chi^{(l)}$, defined by $d\chi^{(l+1)} = d_R\chi^{(l)}$, are in involution with respect to the Poisson brackets defined by P_0 (corresponding with ω) and RP_0.*

From

$$\begin{aligned}\{\chi^{(l)}, g\}_1 &= \omega_0(X_{\chi^{(l)}}, RX_g) = -\langle R(X_g), d\chi^{(l)}\rangle \\ &= -\langle X_g, d_R\chi^{(l)}\rangle = -\langle X_g, d\chi^{(l+1)}\rangle = \{\chi^{(l+1)}, g\}_0 \qquad \forall g \in C^\infty(M),\end{aligned}$$

follows that the inductive definition of the functions $\chi^{(l)}$ can also be expressed as

$$\{\chi^{(l+1)}, \cdot\}_0 = \{\chi^{(l)}, \cdot\}_1.$$

These relations are called the *Lenard recursion relations*.

For completeness we recall here also the definition of the more general concept of a gauged bi-differential calculus.

Definition 1.24. *Suppose that we have a bi-differential calculus (d_1, d_2) on $\Lambda(M)$. A gauged bi-differential calculus on $\Lambda(M)$ is a pair of operators (D_1, D_2) of the form $D_i = d_i + A_i$, where the A_i are $N \times N$ matrices of 1-forms which act on $N \times 1$ column vectors of forms by matrix-wedge multiplication. The D_i further have to satisfy the conditions $D_i^2 = 0$ and $[D_1, D_2] = D_1 D_2 + D_2 D_1 = 0$.*

An example in the case that $N = 1$ is $D_1 = d$ and $D_2 = d_R + df$. The condition $D_2^2 = 0$ reduces to $d_R df = 0$ if we assume that the Nijenhuis torsion of R is zero (or if (d, d_R) defines a bi-differential calculus). If f satisfies this condition, we have a gauged bi-differential calculus since $[D_1, D_2] = 0$ follows from the fact that $[d, d_R] = 0$.

1.4 Special conformal Killing tensors

Consider a Riemannian manifold or pseudo-Riemannian manifold (M, g).

Definition 1.25. *A vector field X on M is a Killing vector if $\mathcal{L}_X g = 0$.*

In coordinates this is equivalent to

$$X_{i|j} + X_{j|i} = 0,$$

where $X_i = g_{ij} X^j$ and where the bar denotes covariant differentiation with respect to the Levi-Civita connection of g. This concept can be generalized for tensor fields.

Definition 1.26. *A symmetric type $(0, 2)$ tensor field L is a Killing tensor if the Schouten bracket $[L, g] = 0$ or in coordinates*

$$L_{ij|k} + L_{jk|i} + L_{ki|j} = 0.$$

Definition 1.27. *A symmetric type* $(0,2)$ *tensor field* L *is a conformal Killing tensor if there is a 1-form* α *such that*

$$L_{ij|k} + L_{jk|i} + L_{ki|j} = \alpha_i g_{jk} + \alpha_j g_{ki} + \alpha_k g_{ij}.$$

If α is exact L is called a *conformal Killing tensor of gradient type* and if $\alpha = d(\operatorname{tr} L)$, L is said to be a *conformal Killing tensor of trace type*. Note that the trace (and determinant) of a type (1,1) tensor field is a scalar, so whenever we use trace (or determinant) it is to be assumed that the corresponding tensor field is in type (1,1) form, e.g. $L^i_j := g^{ik} L_{kj}$ where $(g^{ij}) = (g_{ij})^{-1}$. If $\alpha = 0$, L is a Killing tensor.

Definition 1.28. *A symmetric type* $(0,2)$ *tensor field* J *such that*

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik}) \tag{1.10}$$

for some α_i*, is called a special conformal Killing tensor (scKt).*

The equivalent condition for the (1,1) form of the tensor field is,

$$J^i_{j|k} = \tfrac{1}{2}(\alpha_l g^{li} g_{jk} + \alpha_j \delta^i_k). \tag{1.11}$$

It follows that $\alpha = d(\operatorname{tr} J)$.

The term special conformal Killing tensor was probably first used in [19] but scKts for general metrics have already appeared, with slightly different assumptions and under different names, in the work of Benenti. See for example the comprehensive review paper [8], where Benenti calls them *L-tensors* in the case of pointwise simple eigenvalues. Therefore scKts with pointwise simple eigenvalues are also often called *Benenti tensors*. For the case of an Euclidean space they were studied by Benenti [6] and Lundmark [40], where they were called, respectively, *planar inertia tensors* and *elliptic coordinates matrices*. For the Euclidean metric it is easy to solve (1.10): the general solution has the form

$$J_{ij} = a q^i q^j + b_i q^j + b_j q^i + c_{ij}$$

where the a, b_i and $c_{ij} = c_{ji}$ are constants.

Special conformal Killing tensors have a lot of interesting properties.

A first property, which we shall use in Chapter 4, is related to the cofactor tensor of a nonsingular scKt J. Note that the *cofactor tensor* A of a type (1,1) tensor field J (notation $A = \operatorname{cof} J$) is defined by the relation $JA = AJ = (\det J)I$.

Proposition 1.29. *The cofactor tensor of a nonsingular special conformal Killing tensor is a Killing tensor.*

Proof. By taking the covariant derivative of the equation $A_{ij}J^j_l = (\det J)g_{il}$ and using the defining condition (1.11), the result follows. □

Proposition 1.30. *The Nijenhuis torsion N_J of a special conformal Killing tensor J vanishes.*

Proof. The Nijenhuis torsion N_J (1.1) has coefficients

$$(N_J)^k_{ij} = J^k_l(J^l_{i|j} - J^l_{j|i}) - J^l_j J^k_{i|l} + J^l_i J^k_{j|l}$$

with respect to the standard coordinate basis. From (1.11) it then easily follows that $N_J = 0$. □

There are also converse results.

Proposition 1.31. *A conformal Killing tensor of trace type, with vanishing Nijenhuis torsion, is a special conformal Killing tensor.*

Proof. See Theorem A.5.3 in [8]. □

For the last two properties of scKts mentioned in this section we refer to [20]. They are only valid on a (strictly) Riemannian manifold

Proposition 1.32. *A conformal Killing tensor with functionally independent eigenfunctions whose Nijenhuis torsion vanishes is a special conformal Killing tensor.*

Proof. See Proposition 1 in [20]. □

Moreover if a scKt has n functionally independent eigenfunctions λ_i, the $\{\lambda_i\}$ can be taken as local coordinates and, since the Nijenhuis torsion $N_J = 0$ [29], J takes the form

$$J = \lambda_i \frac{\partial}{\partial \lambda_i} \otimes d\lambda_i$$

with respect to them.

Another interesting theorem in [20] concerns manifolds which admit two or more scKts.

Theorem 1.33. *If a Riemannian manifold admits two scKts L and M which are independent, in the sense that they have no nontrivial common invariant subspaces, and if further L has functionally independent eigenfunctions, the manifold is a space of constant curvature.*

Special conformal Killing tensors play an important role in several areas of research. In the next section we will for example discuss the role of scKts in the study of separability of the Hamilton-Jacobi equation. Also in the definition of a certain class of completely integrable dynamical systems, namely (driven) cofactor systems, scKts are involved. We will discuss this in detail in Chapter 4.

1.5 Separability of the Hamilton-Jacobi equation

In this section we give a brief overview of some (historical) developments in the area of separation of variables. Suppose that we have a time-dependent Hamiltonian $H(t, q, p)$. The associated *Hamilton-Jacobi equation* is given by

$$H\left(t, q, \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0. \tag{1.12}$$

This is a first-order partial differential equation for $S(t, q)$. A complete integral $S(t, q, \alpha)$ of the Hamilton-Jacobi equation is a solution of (1.12) depending on $n+1$ independent constants. Since in (1.12) only derivatives of S appear, the complete integral is defined up to an additive constant, α_0, that may be ignored and let α represent the remaining n independent constants $\alpha_1, \ldots, \alpha_n$. Moreover, a complete integral needs to satisfy the regularity condition

$$\det\left(\frac{\partial^2 S}{\partial q^i\, \partial \alpha_j}\right) \neq 0.$$

An important result, proved by Jacobi, is that the solution of Hamilton's equations follows from the knowledge of a complete integral of the Hamilton-Jacobi equation without further integration.

In general (1.12) is very hard to solve. In certain cases, however, we can use separation of variables, which is a very powerful solution method. This means that one tries to find a complete solution of the form

$$S(t, q, \alpha) = S_0(t, \alpha) + \sum_{i=1}^{n} S_i(q^i, \alpha).$$

If such a solution exists, we say that the Hamilton-Jacobi equation can be solved by *separation of variables*. H is then said to be *separable* and the $S_i(q^i, \alpha)$ can in principle be determined by quadratures.

In the specific case where the Hamiltonian is not time-dependent (autonomous Hamiltonian systems), one can suggest a solution of the form

$$S(t, q, \alpha) = W(q, \alpha) - \alpha_1 t.$$

The Hamilton-Jacobi equation then reduces to

$$H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_1.$$

Then the Hamiltonian is separable if a solution exists of the form

$$W(q, \alpha) = \sum_{i=1}^{n} W_i(q^i, \alpha).$$

The importance of identifying Hamiltonian systems which are separable was recognized soon after the Hamilton-Jacobi equation was derived. We give here a number of interesting results concerning separation of variables in the context of autonomous Hamiltonian systems. For the study of separability for time-dependent systems we refer to Chapter 3.

1.5.1 Coordinate dependent conditions for separability

Liouville [39] was the first to study the separability of the Hamilton-Jacobi equation. Around 1846 he proved that for $n = 2$, systems with a Hamiltonian of the form

$$H(q, p) = \frac{1}{2}\left(C_1(q)p_1^2 + C_2(q)p_2^2\right) + V(q), \tag{1.13}$$

where the C_k are positive functions, are separable if and only if

$$\begin{aligned} C_k &= \frac{a_k(q^k)}{c}, \\ V &= \frac{V_1(q^1) + V_2(q^2)}{c}, \\ c &= c_1(q^1) + c_2(q^2). \end{aligned}$$

Note that the form of H indicates that the underlying coordinate system is orthogonal since there are no cross product terms of the form $p_k p_i$ in (1.13). The Hamiltonian is then said to be separable in *orthogonal coordinates*.

In 1891, Stäckel generalized the result of Liouville for $n \geq 3$.

Definition 1.34. *A Stäckel matrix is a regular $n \times n$ matrix $(\psi_i^{(j)})$, such that the i-th row consists of functions of q^i only: $\psi_i^{(j)} = \psi_i^{(j)}(q^i)$.*

Denote the inverse of $(\psi_i^{(j)})$ by $(\psi^k_{(l)})$, so that $\psi^k_{(l)}\psi_k^{(j)} = \delta_l^j$ and $\psi^j_{(k)}\psi_l^{(k)} = \delta_l^j$.

Theorem 1.35. *[59] A Hamiltonian of the form $H = \frac{1}{2}\sum_{k=1}^n C_k(q)p_k^2$ is separable if and only if there exists an invertible $n \times n$ Stäckel matrix $\psi_i^{(j)}$ such that*

$$\sum_{k=1}^n C_k \psi_k^{(1)} = 1,$$
$$\sum_{k=1}^n C_k \psi_k^{(j)} = 0, \qquad j = 2, \dots, n.$$

If there is an extra potential term $V(q)$, it should be of the form $V(q) = \sum_{k=1}^n \eta_k(q^k)C_k$ for some functions $\eta_k(q^k)$.

A Hamiltonian system that satisfies the conditions of the theorem of Stäckel is called a *Stäckel (separable) system*. Remark that it follows that $[C_1, \dots, C_n]$ is the first row of $(\psi^i_{(j)})$, the inverse of $(\psi_i^{(j)})$.

The results of Liouville and Stäckel apply to Hamiltonians where the underlying coordinate system is orthogonal. In 1904, Levi-Civita [37] found a test for the separability of a given Hamiltonian system, not necessarily in orthogonal coordinates.

Theorem 1.36. *The Hamilton-Jacobi equation associated with a general Hamiltonian H separates in the coordinates (q^i, p_i) if and only if*

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial^2 H}{\partial q^i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial q^i\,\partial p_j}\frac{\partial H}{\partial q^j}\right) = \frac{\partial H}{\partial q^i}\left(\frac{\partial^2 H}{\partial p_i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial p_i\,\partial p_j}\frac{\partial H}{\partial q^j}\right), \tag{1.14}$$

$i, j = 1, \dots, n$ and $i \neq j$ (there is no summation over repeated indices).

Levi-Civita also proved that if $H = T + V$ is separable, also T is separable in the same coordinates. Thus the separation of the geodesic Hamiltonian $T = \frac{1}{2}g^{ij}p_i p_j$ is a necessary condition for the separation of the complete Hamiltonian $H = T + V$. This is why the geodesic Hamiltonian merits primary attention in a lot of results.

Remark that for time-dependent Hamiltonian systems Forbat [30] generalized in 1944 these conditions. The conditions of Forbat are in fact the Levi-Civita conditions plus another set of conditions involving in addition partial derivatives of H with respect to t. In Chapter 3 we will discuss them in detail and give a geometric description.

The Levi-Civita conditions are evidently fundamental: they provide a straightforward test for separability. However, they suffer from a major disadvantage, namely they don't tell you how to construct separation coordinates or if they even exist. The first theorem that tries to give an answer to this is the result of Eisenhart.

1.5.2 Intrinsic conditions for separability

In 1934, Eisenhart related the orthogonal separability of a Riemannian metric to the existence of first integrals of the Hamiltonian system quadratic in the momenta [26].

Theorem 1.37. *The geodesic Hamiltonian $H = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}(q)p_i p_j$ is orthogonally separable if and only if the following conditions are satisfied*

1. *there exist n linearly independent (2,0) Killing tensors $K_{(m)}$ for $m = 1, \ldots, n$ with $K_{(1)} = g$ ($K^{ij}_{(m)} = g^{ik}g^{jl}K_{(m)kl}$),*
2. *the quadratic functions $H_{(m)} = \frac{1}{2}\sum_{i,j=1}^{n} K^{ij}_{(m)} p_i p_j$ commute pairwise with respect to the standard Poisson bracket,*
3. *each $K_{(m)}$ $(m > 1)$ has n functionally independent eigenfunctions $\lambda^k_{(m)}$ and the $n \times n$ matrix whose rows are the eigenfunctions of $K_{(m)}$ for $m = 1, \ldots, n$ is everywhere nonsingular,*
4. *the $K_{(m)}$ are simultaneously diagonalizable and one can find a set of simultaneous eigenforms for them which are closed:*

$$\sum_{i=1}^{n} \left(K^{ij}_{(m)} - \lambda^k_{(m)} g^{ij} \right) \alpha^k_i = 0 \quad \text{with} \quad d\alpha^k = 0.$$

Remark that conditions 1 and 2 imply that the $H_{(m)}$ $(m > 1)$ are $(n-1)$ independent first integrals. The closed eigenforms determine an orthogonal coordinate system y^k (with $dy^k = \alpha^k$) with respect to which the tensors $K_{(m)}$ are all diagonal. A set of tensors $K_{(m)}$ satisfying the conditions of Eisenhart's Theorem is called a *Killing-Stäckel system* [8].

A standard work on the classification of coordinate systems in which the Hamilton-Jacobi equation separates for Riemannian spaces of constant curvature is the monograph of Kalnins [33].

Also Benenti has contributed a lot to the evolution of the subject in the past decades, see for example [5], [6] and [7]. We mention some of his important results.

Theorem 1.38. *A Hamiltonian $H = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij}(q)p_ip_j$ is separable in orthogonal coordinates if and only if there exists a Killing tensor K for the kinetic energy metric g^{ij}, with pointwise real simple eigenvalues and orthogonally integrable eigenvectors (or closed eigenforms). Admissible potentials $V(q)$ should satisfy the corresponding condition $d(KdV) = 0$.*

In this case, the tensor is called a *characteristic Killing tensor*.

Theorem 1.39. *If a Riemannian manifold (M, g) admits a scKt J with functionally independent eigenfunctions λ_i, $i = 1, \ldots, n$, the Hamiltonian $H = \frac{1}{2}g^{ij}p_ip_j$ is separable in the orthogonal coordinates λ_i.*

The proof is in fact a brute calculation showing that the Levi-Civita conditions are satisfied, see [20].

Theorem 1.40. *[9] Let L be a symmetric type $(0, 2)$ tensor field with eigenvalues (u^i). The set of tensors $\{K_{(a)}\} = \{K_{(0)}, \ldots, K_{(n-1)}\}$ defined by*

$$K_{(0)} = g, \qquad K_{(a)} = \frac{1}{a}\,\mathrm{tr}(K_{(a-1)}L)g - K_{(a-1)}L \qquad a > 0 \tag{1.15}$$

defines a basis of a Killing-Stäckel system if and only if L is a conformal Killing tensor with vanishing Nijenhuis torsion and pointwise simple eigenvalues (called an L-tensor in the work of Benenti).

The tensors in (1.15) can also be defined by [9]

$$K_{(a)} = \sum_{k=0}^{a}(-1)^k\sigma_{a-k}L^k, \quad \text{or} \quad K_{(a)} = \sigma_a g - K_{(a-1)}L \ \text{ with } \ K_{(-1)} = 0,$$

where σ_a is the elementary symmetric polynomial of order a of the eigenvalues u^i of L. Recall that the elementary symmetric polynomials are defined by the identity

$$\prod_{i=1}^{n}(z+u^i)=\sum_{k=0}^{n}\sigma_k(u)z^{n-k}. \tag{1.16}$$

Note that in applying these formulas we need to know the eigenvalues of L. This makes it less effective then (1.15) where the Killing tensors are determined in a purely algebraic way starting from the tensor L.

An important remark here is that a scKt with functionally independent eigenfunctions is an L-tensor [8]. So there is a direct link between Theorem 1.38 and Theorem 1.39. Given a scKt J with functionally independent eigenfunctions, the Killing tensor K defined by $K=\mathrm{tr}(J)g-J$ is a characteristic Killing tensor. Note that K is in fact $K_{(1)}$ in the basis of the Killing-Stäckel system in Theorem 1.40. For testing if an additional potential is separable, one can apply Theorem 1.38 by using the Killing tensor K.

There is also another interesting link between the theorem of Eisenhart (Theorem 1.37) and scKts. Suppose that J is a scKt, then so is $J+aI$ for any constant a. Let $A(a)$ denote the cofactor tensor of $J+aI$, it is then a Killing tensor for every a. Define the tensors $A_{(i)}$ by

$$A(a)=\sum_{i=1}^{n}A_{(i)}a^{n-i}.$$

Theorem 1.41. *If a special conformal Killing tensor J has functionally independent eigenfunctions, the tensors $A_{(i)}$ $(i=1,\dots,n)$ form a Killing-Stäckel system.*

Proof. See Theorem 1 in [20]. □

If one has a scKt available, generating the full Stäckel system is a purely algebraic process. Furthermore the separation coordinates are simply the eigenfunctions of the scKt.

In fact it follows from the defining equation $A(a)(J+aI)=\det(J+aI)I$ and the property $\det(J+aI)=\prod_{i=1}^{n}(u^i+a)$ with u^i the eigenfunctions of J, that the $A_{(i)}$ are defined by the recurrence relation

$$A_{(i+1)}=-A_{(i)}J+\sigma_iI, \qquad A_{(1)}=I,$$

so $A_{(i)}$ corresponds to $K_{(i-1)}$ in (1.15).

An intrinsic way of writing down the Levi-Civita conditions (1.14) was suggested by Magri and inspired Crampin [21]. He stated that if the Hamiltonian $H = \frac{1}{2}g^{ij}p_i p_j$ is separable in orthogonal coordinates (x^i), then for every symmetric type (1,1) tensor field J which is diagonal with respect to the coordinates (x^i), has pointwise simple eigenvalues and vanishing Nijenhuis torsion,

$$dd_{\widetilde{J}}H = \sum_{r=0}^{n-1} \alpha^{(r)} \wedge d_{\widetilde{J}^r}H \tag{1.17}$$

with $\widetilde{J}$ the complete lift of J to T^*M and $\alpha^{(r)}$ 1-forms on M. Conversely, if there is a locally defined type (1,1) tensor field J which is symmetric with respect to g, has pointwise simple eigenvalues and vanishing Nijenhuis torsion and if there exist 1-forms $\alpha^{(r)}$ such that $\widetilde{J}$ satisfies (1.17), then any coordinates with respect to which J is diagonal are orthogonal separation coordinates for H. The special case

$$dd_{\widetilde{J}}H = \alpha \wedge dH$$

implies that J is symmetric with respect to g and moreover satisfies (1.10); in other words, J is a scKt. So if a metric g admits a scKt it is separable. If the eigenfunctions of the scKt are functionally independent then they are separation coordinates; otherwise one has to find coordinates with respect to which J is diagonal. This is in accordance with Theorem 1.39.

To conclude this chapter, we discuss the role of bi-Hamiltonian manifolds in the theory of separability for integrable systems. The papers of Falqui and Pedroni, [27] and [28], are good references in this context. Recall that a Hamiltonian H is called *(completely) integrable* if, along with $H = H_1$, we have $n-1$ functionally independent, pairwise commuting integrals of motion $H_2, \ldots, H_n$. An integrable system is separable if the Hamilton-Jacobi equation associated with any H_i is separable. An equivalent definition is the following.

Definition 1.42. *[58] An integrable system $(H_1, \ldots, H_n)$ is separable in the coordinates (q, p) if there exist n relations, called separation relations, of the form*

$$\phi_i(q^i, p_i, H_1, \ldots, H_n) = 0, \qquad \det\left[\frac{\partial \phi_i}{\partial H_j}\right] \neq 0, \tag{1.18}$$

for $i = 1, \ldots, n$.

In this case, an intrinsic test of separability can be expressed using ωN-manifolds. Consider a semisimple ωN-manifold, then the following statements are equivalent:

1. the integrable system $(H_1, \dots, H_n)$ is separable in the Darboux-Nijenhuis coordinates;

2. the functions $H_1, \dots, H_n$ are in involution with respect to both Poisson brackets:

$$\{H_i, H_j\}_0 = \{H_i, H_j\}_1 = 0, \qquad \forall\, i, j;$$

3. there exists a simple matrix $F = (F_{ij})$, called the control matrix, such that

$$N dH_i = \sum_{i=1}^{n} F_{ij} dH_j, \qquad i = 1, \dots, n. \tag{1.19}$$

   Moreover, F has the same eigenvalues as N and the relations (1.19) are called generalized Lenard relations.

An integrable system is called Stäckel separable if the separation relations (1.18) are affine in the H_j, that is,

$$\phi_i(q^i, p_i, H_1, \dots, H_n) = \sum_{j=1}^{n} S_{ij}(q^i, p_i) H_j - U_i(q^i, p_i). \tag{1.20}$$

Then, an integrable system is Stäckel separable if and only if in addition to (1.19), the matrix (F_{ij}) satisfies

$$N dF_{ij} = \sum_{k=1}^{n} F_{ik} dF_{kj}, \qquad \forall\, i, j = 1, \dots, n.$$

The matrix (S_{ij}) in (1.20) can be shown to be a suitably normalized matrix that diagonalizes the matrix (F_{ij}):

$$F_{ij} = \sum_{k=1}^{n} S^{-1}_{ik} \lambda_k S_{jk}.$$

Its characteristic property is that the entries S_{ij} of the i-th row depend only on the pair (q^i, p_i) of Darboux-Nijenhuis coordinates. That is why the matrix (S_{ij}) is also called a Stäckel matrix.

CHAPTER

2

LIFTING GEOMETRIC OBJECTS TO $J^1\tau^*$

Starting from geometric objects on a bundle $\tau : E \to \mathbb{R}$, corresponding lifted objects on TE, T^*E or $J^1\tau$ are rather well known. Less attention is paid to the case of lifting operations to the dual of the first jet bundle $J^1\tau^*$, with the exception of [31]. In this chapter, we discuss various lifts of vector fields and 1-forms to $J^1\tau^*$ and list some of their immediate properties. The main objective is to come to an intrinsic definition of the complete lift of a type (1,1) tensor field from E to $J^1\tau^*$ and to study its properties. They are indispensable for proving that the canonical Poisson structure on $J^1\tau^*$, together with the complete lift of a type (1,1) tensor field R on E with vanishing Nijenhuis torsion, determine a Poisson-Nijenhuis structure on $J^1\tau^*$. The construction of Darboux-Nijenhuis coordinates for this structure is explained in detail.

2.1 The first jet bundle and its dual: $J^1\tau$ and $J^1\tau^*$

In this section we introduce the first jet bundle $J^1\tau$ and the dual of the first jet bundle $J^1\tau^*$ of a bundle $\tau : E \to \mathbb{R}$. These spaces are for us, as we will argue in the next section, the spaces to be for the analysis of intrinsic aspects of time-dependent Lagrangian and Hamiltonian systems. We will restrict ourselves to the most relevant definitions, based on [57] and [16]. We refer to these books for more details.

Let us first recapitulate some general definitions related to fibre bundles. A manifold E is fibred over a manifold M if there exists a smooth map $\pi : E \to M$ which is a surjective submersion. The manifold E is called *the total space*, π *the projection* and M *the base space*. A fibred manifold is called *locally trivial* if there exists a manifold S, and if each point p in M admits an open neighbourhood $U \subset M$ and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times S$ such that $pr_1 \circ \psi = \pi$, meaning that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times S \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

A locally trivial fibred manifold is called a *fibre bundle*, or shortly, a bundle. The manifold S is called *the standard fibre* of the fibre bundle and for each point $p \in M$, the subset $E_p = \pi^{-1}(p) \subset E$ is called the fibre over p.

Fix a point p in M. Consider (U, ψ) around p as in the definition above, and the fibre E_p. Since all elements in E_p project, via π, onto p, the diffeomorphism ψ restricts to a diffeomorphism

$$\psi_p = pr_2 \circ \psi|_{E_p} : E_p \to S.$$

between E_p and S. A local trivialization allows one to introduce a special atlas on the total space E consisting of charts which we call 'adapted to the fibre bundle structure'. Let V be a coordinate chart on M which, possibly after restriction, belongs to a trivializing neighbourhood (U, ψ), and let W be a coordinate chart on S. Then $\psi^{-1}(V \times W)$ defines a coordinate chart on E and the coordinates of a point m in this chart may be taken to be $(x_i; u^\alpha)$, where (x_i) are the coordinates of $\pi(m) = p \in V$ and (u^α) are the coordinates of $\psi_p(m) \in W$. Coordinates constructed as above are said to be adapted to the fibration.

The *trivial bundle* or product bundle with base M and standard fibre S is the fibre bundle $pr_1 : M \times S \to M$. A fibre bundle $\pi : E \to M$ is called *trivializable* if it admits a global trivialization, i.e. there exists a (global) diffeomorphism $\psi : E \to M \times S$ such that $pr_1 \circ \psi = \pi$. It is a general property that any fibre bundle over a contractible base M is trivializable [1].

A section ϕ of a bundle is a smooth map $\phi : M \to E$ satisfying $\pi \circ \phi = Id_M$.

We will now give two special classes of fibre bundles. To do so, we will specify the structure of the standard fibre and we will require the maps ψ to satisfy extra properties.

Definition 2.1. *A vector bundle is a bundle $\pi : E \to M$ such that*

1) *for each $p \in M$, the fibre E_p is a vector space of a fixed dimension n;*
2) *the standard fibre is $\mathbb{R}^n$;*
3) *the local trivialization $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ of the fibre bundle π is linear, that is to say: for each $q \in U$, the map $\psi_q : E_q \to \mathbb{R}^n$ is a linear isomorphism.*

A well known example of a vector bundle is the tangent bundle of a differentiable manifold.

Definition 2.2. *A fibre bundle $\pi : E \to M$ is an affine bundle modelled on the vector bundle $\tau : F \to M$ if*

1) *for each $p \in M$, the fibre E_p is an affine space modelled on the vector space F_p;*
2) *the standard fibre is $\mathbb{R}^n$, where n is the fibre dimension of $\tau : F \to M$;*
3) *the local trivialization $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ of the fibre bundle π is affine, that is to say: for each $q \in U$, the map $\psi_q : E_q \to \mathbb{R}^n$ is an affine isomorphism from the affine space E_q to the affine space $\mathbb{R}^n$, whose linear part is the linear isomorphism from F_q to $\mathbb{R}^n$ corresponding to a linear local trivialization of the vector bundle $\tau : F \to M$.*

We now have enough background to define the first jet bundle. Consider a bundle $\tau : E \to \mathbb{R}$ with $\dim E = n+1$ and local adapted coordinates denoted by (t, q^i). The projection map induces a linear map of tangent spaces $\tau_* : T_xE \to T_{\tau(x)}\mathbb{R}$, whose kernel is the subspace of vectors tangent to the fibre through $x = (t, q)$. This space is called the vertical subspace at $x = (t, q)$, denoted by $V_x\tau$. Note that the vertical vectors form a sub-bundle of TE and denote it $V\tau$.

Two local sections ϕ, ψ of τ are said to be 1-equivalent at $t' \in \mathbb{R}$ with $t' \in \text{dom } \phi \cap \text{dom } \psi$, if

$$\phi^i(t') = \psi^i(t') \quad \text{and} \quad \left.\frac{d\phi^i(t)}{dt}\right|_{t=t'} = \left.\frac{d\psi^i(t)}{dt}\right|_{t=t'}$$

where $\phi^i = q^i \circ \phi$ and $\psi^i = q^i \circ \psi$ for $i = 1, \ldots, n$. One can check that this defines an equivalence relation on the set of local sections of τ. The equivalence class containing ϕ is called the 1-*jet* of ϕ at t' and is denoted $j^1_{t'}\phi$.

Proposition 2.3. *The set*

$$J^1\tau := \{j^1_{t'}\phi \,|\, t' \in \mathbb{R}, \phi \textit{ local section of } \tau \textit{ with } t' \in \textit{dom } \phi\}$$

has the structure of a $(2n + 1)$*-dimensional differentiable manifold and is called the first jet manifold of* $\tau : E \to \mathbb{R}$.

Two projections can be defined: the *source projection*

$$\tau_0 : J^1\tau \to \mathbb{R} : j^1_t\phi \to t$$

and the *target projection*

$$\tau_1 : J^1\tau \to E : j^1_t\phi \to \phi(t).$$

Based on the adapted coordinate system (t, q^i) of E, the induced coordinate system on $J^1\tau$ is $(t, q^i, \dot{q}^i)$, where t is the coordinate of the source of $j^1_t\phi$, q^i are the fibre coordinates of $x = \phi(t)$, the target of $j^1_t\phi$, and $\dot{q}^i = (d\phi^i/dt)\big|_t$.

Proposition 2.4. *The source projection* $\tau_0 : J^1\tau \to \mathbb{R}$ *gives* $J^1\tau$ *the structure of a fibre bundle over* $\mathbb{R}$*. The target projection* $\tau_1 : J^1\tau \to E$ *gives* $J^1\tau$ *the structure of an affine bundle over* E*, called the first jet bundle, modelled on the vertical sub-bundle* $V\tau$ *of* TE*.*

Note that $J^1\tau$ is an affine sub-bundle of TE, locally determined by the constraint $\dot{t} = 1$ on the natural bundle coordinates $(t, q^i, \dot{t}, \dot{q}^i)$ on TE.

Every fibre $J^1_x\tau$ ($\dim J^1_x\tau = n$) of the affine bundle $\tau_1 : J^1\tau \to E$ over $x \in E$ has a corresponding extended dual $(J^1_x\tau)^\dagger$, defined as the $(n+1)$-dimensional vector space of real valued affine functions on $J^1_x\tau$. In this case $(J^1_x\tau)^\dagger \cong T^*_xE$. The dual of $J^1_x\tau$, denoted by $J^1_x\tau^*$, is then the quotient of the space of affine functions by the constant functions and may be identified with the space $T^*_xE/\langle dt\rangle$. So we come to the following definition of the affine dual of the first jet bundle.

Definition 2.5. *The affine dual bundle of the first jet bundle $\tau_1 : J^1\tau \to E$, sometimes also called the vertical cotangent bundle and denoted by $\pi : J^1\tau^* \to E$, is the quotient bundle $T^*E/\langle dt\rangle \to E$.*

We call this bundle for short the dual of the first jet bundle.

There are natural projections $\rho : T^*E \to J^1\tau^*$ and $\pi : J^1\tau^* \to E$. Each point $m \in J^1\tau^*$ is an equivalence class $\langle\alpha\rangle$ of covectors $\alpha \bmod dt$ at $\pi(m)$; saying that m has coordinates (t, q^i, p_i) means that a representative of the class is given by $\alpha_{(t,q)} = p_i dq^i$. Elements of $J^1\tau^*$ have a well-defined action on vertical tangent vectors to E:

$$\text{for } \; v = v^i \frac{\partial}{\partial q^i}\bigg|_{(t,q)} \in T_{(t,q)}E, \quad \text{we have } \langle v, \langle\alpha\rangle\rangle = v^i p_i.$$

Now, let us go deeper into the geometry of $J^1\tau^*$. Unlike T^*E, $J^1\tau^*$ does not carry a canonical 1-form, but there is a canonical equivalence class of 1-forms modulo the module generated by dt, which we shall denote by $\langle\theta\rangle$. As an element of $\mathcal{X}^*(J^1\tau^*)/\langle dt\rangle$, it is meant to have a well-defined action, at each point $m \in J^1\tau^*$, on vectors which are vertical with respect to the projection $\tau \circ \pi : J^1\tau^* \to \mathbb{R}$.

Definition 2.6. *The equivalence class $\langle\theta\rangle \in \mathcal{X}^*(J^1\tau^*)/\langle dt\rangle$ is defined as follows: $\forall m \in J^1\tau^*$ and $v_m \in T_m(J^1\tau^*)$, vertical with respect to $\tau \circ \pi$, we have*

$$\langle v_m, \langle\theta\rangle_m\rangle = \langle T\pi(v_m), m\rangle.$$

In coordinates, $\langle\theta\rangle = (p_i dq^i) \bmod dt$. It follows that

$$\Theta = p_i dq^i \wedge dt \tag{2.1}$$

is a canonically defined 2-form on $J^1\tau^*$. It can be characterized alternatively by the following property, which mimics a well known characterization of the canonical 1-form on a cotangent bundle. A section of the bundle $\pi : J^1\tau^* \to E$ is an element

$\langle\alpha\rangle \in \mathcal{X}^*(E)/\langle dt\rangle$ and for each representative α of the class we have that $\alpha^*\Theta = \alpha \wedge dt$.

On $J^1\tau^*$, being odd dimensional, there exists no symplectic structure. Nevertheless, $J^1\tau^*$ carries a canonical Poisson structure. To show this, consider a general result derived in [38]: given a Poisson manifold (M, Λ) and surjective submersion $\phi : M \to N$, there exists a unique Poisson structure on N if for every pair (f, g) of functions on N and every $m \in N$, the restriction of the Poisson bracket $\{\phi^* f, \phi^* g\}$ to $\phi^{-1}(m)$ is constant. Now, $\rho : T^*E \to J^1\tau^*$ is a surjective submersion and the Poisson bracket of functions ρ^*F and ρ^*G, with $F, G \in C^\infty(J^1\tau^*)$, is a function of the same type, constant on fibres. It follows that $J^1\tau^*$ carries a canonical Poisson structure, which will be denoted by Λ, namely the structure inherited from the Poisson structure on T^*E via the projection $\rho : T^*E \to J^1\tau^*$. In coordinates, we write

$$\Lambda(dF, dG) = \{F(t, q, p), G(t, q, p)\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i},$$

i.e. our sign convention is such that

$$\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}. \tag{2.2}$$

The corresponding Poisson map $P : \mathcal{X}^*(J^1\tau^*) \to \mathcal{X}(J^1\tau^*)$ is defined by $\Lambda(\alpha, \beta) = \langle P(\alpha), \beta\rangle$, and we put

$$X_F = -P(dF) = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{2.3}$$

2.2 Time-dependent systems: generalities

In many papers on time-dependent mechanical systems (the list of citations we could insert here is endless, but to name just a few, in different contexts, see [2], [14], [17], [25], [4]), the trivial bundle $\mathbb{R} \times M$ is chosen as model for the configuration space from the outset. A more sophisticated argument, however, is that for a bundle such as $\tau : E \to \mathbb{R}$ the total space E can be identified with a product of the form $\mathbb{R} \times M$ by choosing a global trivialization. This follows from the more general fact that every fibre bundle over a contractible base is trivializable [1]. It is important to note that requiring a bundle to be trivializable does not give its total space the structure of a Cartesian product in any particular way, i.e. there is no canonical trivialization.

Actually, the identification of E with $\mathbb{R}\times M$ is not ideal for our purposes. The reason for this is that we wish to allow for time-dependent coordinate transformations, since it is the essence of applications in time-dependent mechanics: that is, coordinate transformations on E of the form $(t,x^i)\mapsto(t,y^i)$ with $y^i=y^i(t,x^j)$, together with the induced transformations on T^*E and $J^1\tau^*$. Such coordinate transformations do not respect the product structure which, by implication, has been picked out once and for all. So instead we shall develop the theory for an $(n+1)$-dimensional manifold E which is a fibre bundle over $\mathbb{R}$, with standard fibre M. Although E is trivializable no particular trivialization is to be preferred above any other, and the notation reflects this.

Our goal in this chapter specifically is to study natural lifting operations from tensor fields on E to corresponding tensor fields on $J^1\tau^*$. Therefore, by working in this way we shall also ensure that we don't introduce lifting operations which are natural only under transformations which preserve the corresponding product structure. In this respect, our approach closely relates to the work of Sardanashvily and co-workers on time-dependent mechanics, see [52] and [31]. Other similar settings can be found e.g. in [48].

To conclude this section, note that a time-dependent first-order dynamical system on E is represented by a vector field X on E, which has the property $\langle X,dt\rangle=1$ and is locally of the form

$$X=\frac{\partial}{\partial t}+X^i(t,q)\frac{\partial}{\partial q^i}.$$

2.3 Lifting geometric objects to the cotangent bundle

In this section we recall the main lifts of geometric objects on a manifold M to its cotangent bundle T^*M, based on [15] since it served as a source of inspiration for our work. For some other interesting background information about general aspects of natural operations, see [34]. Another standard reference for lifting operations is [64].

Let us, for this section, denote the canonical 1-form on T^*M as $\theta=p_idq^i$ and let $\pi:T^*M\to M$ be the cotangent bundle projection.

Every vector field on T^*M is completely determined by its action on basic functions and fibre linear functions. So to be able to define various lifts of geometric objects on M to T^*M, we first need the definition of a fibre linear function on T^*M which can be related to a vector field on M.

Definition 2.7. *If $X \in \mathcal{X}(M)$, the fibre linear function h_X on T^*M is the function which at each point $m \in T^*M$ takes the value $h_X(m) = \langle X_{\pi(m)}, m\rangle$.*

If m has coordinates (q,p) and $X = X^i(q)\partial/\partial q^i$, the coordinate expression of h_X is given by $h_X(q,p) = p_i X^i(q)$.

We can recall now the following well-known lift constructions.

Definition 2.8. *Let $X \in \mathcal{X}(M)$, the complete lift of X is the unique vector field $\widetilde{X}$ on T^*M which is π-related with X and is such that $\mathcal{L}_{\widetilde{X}}\theta = 0$.*

Remember that two vector fields $X \in \mathcal{X}(M)$ and $Y \in \mathcal{X}(T^*M)$ are π-related if

$$T\pi \circ Y = X \circ \pi. \tag{2.4}$$

For $X = X^i(q)\partial/\partial q^i$, the complete lift is given by

$$\widetilde{X} = X^i \frac{\partial}{\partial q^i} - p_j \frac{\partial X^j}{\partial q^i}\frac{\partial}{\partial p_i}.$$

Definition 2.9. *For $\alpha \in \mathcal{X}^*(M)$, the vertical lift of α is the π-vertical vector field $\alpha^v \in \mathcal{X}(T^*M)$ which is defined by*

$$\begin{aligned} \alpha^v(\pi^* f) &= 0 \quad \forall f \in C^\infty(M), \\ \alpha^v(h_X) &= \pi^*\langle \alpha, X\rangle \quad \forall X \in \mathcal{X}(M). \end{aligned}$$

In coordinates, if $\alpha = \alpha_i(q)dq^i$,

$$\alpha^v = \alpha_i \frac{\partial}{\partial p_i}.$$

Apart from vector fields and 1-forms, also (1,1) tensor fields can be lifted to T^*M. Consider a (1,1) tensor field R on M:

$$R = R^i_j(q)\frac{\partial}{\partial q^i} \otimes dq^j.$$

There are two different ways of lifting a (1,1) tensor field to T^*M: the first one results in a vertical vector field, the second one in a (1,1) tensor field.

Definition 2.10. *The vertical lift R^v of a (1,1) tensor field R on M is the vertical vector field on T^*M satisfying*

$$R^v(\pi^* f) = 0 \quad \forall f \in C^\infty(M),$$
$$R^v(h_X) = h_{R(X)} \quad \forall X \in \mathcal{X}(M).$$

In coordinates,

$$R^v = p_i R^i_j \frac{\partial}{\partial p_j}.$$

In [15], the complete lift of a (1,1) tensor field R is defined making use of the fibre linear map determined by the (1,1) tensor field:

$$\tau_R : T^*M \to T^*M, (q^i, p_i) \mapsto (q^i, R^i_j p_i).$$

Definition 2.11. *Let R be a (1,1) tensor field on M. The complete lift $\widetilde{R}$ of R to T^*M is the (1,1) tensor field defined by*

$$i_{\widetilde{R}(\xi)} d\theta = i_\xi (\tau_R^* d\theta) \qquad \xi \in \mathcal{X}(T^*M). \tag{2.5}$$

An alternative definition is:

$$i_{\widetilde{R}(\xi)} d\theta = i_\xi (\mathcal{L}_{R^v} d\theta). \tag{2.6}$$

Equivalently $\widetilde{R}$ can be specified explicitly by evaluating it on the vertical lift of a 1-form and on the complete lift of a vector field, since they span the vector fields on T^*M. We will formulate this as a theorem.

Theorem 2.12. *For any 1-form α and vector field X on M,*

$$\widetilde{R}(\alpha^v) = R(\alpha)^v$$
$$\widetilde{R}(\widetilde{X}) = \widetilde{R(X)} + (\mathcal{L}_X R)^v.$$

In coordinates, the complete lift of the tensor field R reads,

$$\widetilde{R} = R^i_j \left(\frac{\partial}{\partial q^i} \otimes dq^j + \frac{\partial}{\partial p_j} \otimes dp_i \right) + p_i \left(\frac{\partial R^i_j}{\partial q^k} - \frac{\partial R^i_k}{\partial q^j} \right) \frac{\partial}{\partial p_j} \otimes dq^k. \tag{2.7}$$

The complete lift $\widetilde{R}$ has some interesting properties, we recall the most interesting one. For more details and the proof we refer to [15].

Corollary 2.13. $N_{\widetilde{R}} = 0$ *if and only if* $N_R = 0$.

2.4 Lifting vector fields and 1-forms to $J^1\tau^*$

In this section we discuss various lifts of vector fields and 1-forms to $J^1\tau^*$, the dual of the first jet bundle of a bundle $\tau : E \to \mathbb{R}$ and list some immediate properties. From now on, we will consider lifts to $J^1\tau^*$. If we make use of a lifting operation to the cotangent bundle T^*E, we will clearly indicate it and use an explicit notational distinction. Recall that we denote the projection of $J^1\tau^*$ to E by π.

The results of [15], recalled in the previous section, served as a source of inspiration for the lifts we define in this section.

Consider $\mathcal{X}_V(E)$, the $C^\infty(E)$-module of vertical vector fields on E. In adapted (t, q) coordinates, a vertical vector field on E is of the form

$$X = X^i(t, q)\frac{\partial}{\partial q^i}.$$

For each $X \in \mathcal{X}_V(E)$ we can define a fibre linear function on $J^1\tau^*$ as follows.

Definition 2.14. *If $X \in \mathcal{X}_V(E)$, we denote by $F_X \in C^\infty(J^1\tau^*)$ the function which at each point $m \in J^1\tau^*$ takes the value $F_X(m) = \langle X_{\pi(m)}, m\rangle$.*

If m has coordinates (t, q, p) and $X = X^i(t, q)\partial/\partial q^i$, the coordinate expression of F_X is given by $F_X(t, q, p) = p_i X^i(t, q)$.

A first set of interesting vector fields on $J^1\tau^*$ is obtained by vertically lifting 1-forms on E.

Definition 2.15. *For $\alpha \in \mathcal{X}^*(E)$, $\alpha^v \in \mathcal{X}(J^1\tau^*)$ is determined by*

$$\begin{aligned} \alpha^v(\pi^* f) &= 0, \quad \forall f \in C^\infty(E) \\ \alpha^v(F_X) &= \pi^*\langle X, \alpha\rangle \quad \forall X \in \mathcal{X}_V(E). \end{aligned}$$

α^v is called the vertical lift of α to $J^1\tau^*$.

In coordinates, if $\alpha = \alpha_0(t, q)dt + \alpha_i(t, q)dq^i$, we have

$$\alpha^v = \alpha_i\frac{\partial}{\partial p_i}. \tag{2.8}$$

It is worth observing that in fact the assignment $\alpha \mapsto \alpha^v$ induces a map from $\mathcal{X}^*(E)/\langle dt\rangle$ into $\mathcal{X}(J^1\tau^*)$.

We define now the complete lift of two classes of vector fields on E, one is the module of vertical vector fields $\mathcal{X}_V(E)$ already introduced; the other is the set of vector fields with the property $\langle X, dt\rangle = 1$ which we shall denote by $\mathcal{X}_t(E)$. Elements of $\mathcal{X}_t(E)$ can be regarded as sections of $J^1\tau \to E$, if $J^1\tau$ is seen as the submanifold of TE described before.

Definition 2.16. *For all $X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$, the complete lift $\widetilde{X} \in \mathcal{X}(J^1\tau^*)$ is defined by the following two requirements:*

1. *$\widetilde{X}$ is π-related to X,*
2. *$\mathcal{L}_{\widetilde{X}}\Theta = 0$, with Θ the canonically defined 2-form on $J^1\tau^*$ (2.1).*

We deduce the coordinate expression of $\widetilde{X}$ for a general $X \in \mathcal{X}_t(E)$, $X = \partial/\partial t + X^i(t,q)\partial/\partial q^i \in \mathcal{X}_t(E)$. Its complete lift is a vector field on $J^1\tau^*$ of the form

$$\widetilde{X} = A(t,q,p)\frac{\partial}{\partial t} + B^i(t,q,p)\frac{\partial}{\partial q^i} + C_i(t,q,p)\frac{\partial}{\partial p_i}.$$

For X and $\widetilde{X}$ to be π-related, we must have that $A = 1$ and $B^i = X^i$. From the second requirement

$$\mathcal{L}_{\widetilde{X}}\Theta = C_i dq^i \wedge dt + p_i\left(\frac{\partial X^i}{\partial t}dt + \frac{\partial X^i}{\partial q^j}dq^j\right) \wedge dt = 0$$

follows that $\widetilde{X}$ is of the form

$$\widetilde{X} = \frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i} - p_j\frac{\partial X^j}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{2.9}$$

Similarly

$$\widetilde{X} = X^i\frac{\partial}{\partial q^i} - p_j\frac{\partial X^j}{\partial q^i}\frac{\partial}{\partial p_i}, \quad \text{for } X = X^i(t,q)\frac{\partial}{\partial q^i} \in \mathcal{X}_V(E). \tag{2.10}$$

Both types of complete lifts are introduced in [31] in a different way, mainly based on coordinate calculations.

It is instructive to verify by direct coordinate calculations that all lifting constructions so far introduced are indeed well defined, i.e. behave properly under a time-dependent coordinate transformation $t = t$, $Q^i = Q^i(t,q)$ on E and induced transformation $(t,q,p) \mapsto (t,Q,P)$ on $J^1\tau^*$, where

$$P_j = p_i\frac{\partial q^i}{\partial Q^j}(t, Q(t,q)).$$

As an example, we check that $\widetilde{X}$ is well defined for $X \in \mathcal{X}_V(E)$. In the new coordinates (t, Q^i) on E, X is of the form

$$X = X^i(t, q(t,Q))\frac{\partial Q^k}{\partial q^i}\frac{\partial}{\partial Q^k} = \bar{X}^k(t,Q)\frac{\partial}{\partial Q^k},$$

its complete lift is

$$\begin{aligned}\widetilde{X} &= \bar{X}^k\frac{\partial}{\partial Q^k} - P_j\frac{\partial \bar{X}^j}{\partial Q^k}\frac{\partial}{\partial P_k}\\ &= X^i\frac{\partial Q^k}{\partial q^i}\frac{\partial}{\partial Q^k} - P_j\frac{\partial X^i}{\partial q^l}\frac{\partial q^l}{\partial Q^k}\frac{\partial Q^j}{\partial q^i}\frac{\partial}{\partial P_k} - P_jX^i\frac{\partial^2 Q^j}{\partial q^i\,\partial q^r}\frac{\partial q^r}{\partial Q^k}\frac{\partial}{\partial P_k}.\end{aligned} \tag{2.11}$$

On the other hand, if we first compute the complete lift of X in the old coordinates (t, q, p) and then apply the induced transformation on $J^1\tau^*$, we get

$$\begin{aligned}\widetilde{X} &= X^i\left(\frac{\partial Q^k}{\partial q^i}\frac{\partial}{\partial Q^k} + p_rX^i\frac{\partial^2 q^r}{\partial Q^k\,\partial Q^l}\frac{\partial Q^l}{\partial q^i}\frac{\partial}{\partial P_k}\right) - P_j\frac{\partial Q^j}{\partial q^i}\frac{\partial X^i}{\partial q^l}\frac{\partial q^l}{\partial Q^k}\frac{\partial}{\partial P_k}\\ &= X^i\frac{\partial Q^k}{\partial q^i}\frac{\partial}{\partial Q^k} + P_j\frac{\partial Q^j}{\partial q^r}X^i\frac{\partial^2 q^r}{\partial Q^k\,\partial Q^l}\frac{\partial Q^l}{\partial q^i}\frac{\partial}{\partial P_k} - P_j\frac{\partial Q^j}{\partial q^i}\frac{\partial X^i}{\partial q^l}\frac{\partial q^l}{\partial Q^k}\frac{\partial}{\partial P_k}\\ &= X^i\frac{\partial Q^k}{\partial q^i}\frac{\partial}{\partial Q^k} - P_jX^i\frac{\partial^2 Q^j}{\partial q^s\,\partial q^r}\frac{\partial q^r}{\partial Q^k}\frac{\partial q^s}{\partial Q^l}\frac{\partial Q^l}{\partial q^i}\frac{\partial}{\partial P_k} - P_j\frac{\partial Q^j}{\partial q^i}\frac{\partial X^i}{\partial q^l}\frac{\partial q^l}{\partial Q^k}\frac{\partial}{\partial P_k}\\ &= X^i\frac{\partial Q^k}{\partial q^i}\frac{\partial}{\partial Q^k} - P_jX^i\frac{\partial^2 Q^j}{\partial q^i\,\partial q^r}\frac{\partial q^r}{\partial Q^k}\frac{\partial}{\partial P_k} - P_j\frac{\partial Q^j}{\partial q^i}\frac{\partial X^i}{\partial q^l}\frac{\partial q^l}{\partial Q^k}\frac{\partial}{\partial P_k}\end{aligned} \tag{2.12}$$

where we make use, in the penultimate step, of the equality following from deriving the identity $Q^j \equiv Q^j(t, q(t,Q))$ with respect to Q^k and Q^l:

$$0 \equiv \frac{\partial^2 Q^j}{\partial q^s\,\partial q^r}\frac{\partial q^r}{\partial Q^k}\frac{\partial q^s}{\partial Q^l} + \frac{\partial Q^j}{\partial q^r}\frac{\partial^2 q^r}{\partial Q^k\,\partial Q^l}.$$

It immediately follows from (2.11) and (2.12) that $\widetilde{X}$ is well defined. By playing with the identities $q^j \equiv q^j(t, Q(t,q))$ and $Q^j \equiv Q^j(t, q(t,Q))$, this can be checked in a similar way for all lifts defined in this chapter.

Some immediate properties of $\widetilde{X}$ are

$$\begin{aligned}&\text{for } X \in \mathcal{X}_V(E), \quad i_{\widetilde{X}}\Theta = p_iX^i dt = F_X dt,\\ &\text{for } X \in \mathcal{X}_t(E), \quad i_{\widetilde{X}}\Theta \wedge dt = (p_iX^i dt - p_i dq^i) \wedge dt = -\Theta.\end{aligned}$$

The main motivation for introducing the vector fields α^v and both types of $\widetilde{X}$, is that together they provide a local basis of vector fields on $J^1\tau^*$. As such, they are perfectly suited to define other tensorial objects on $J^1\tau^*$ in a coordinate free way, as we will see in the subsequent sections. It will then be interesting to have expressions for the Lie brackets of such vector fields.

Lemma 2.17. *For all $\alpha, \beta \in \mathcal{X}^*(E)$ and $X, Y \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$, we have*

$$[\alpha^v, \beta^v] = 0, \tag{2.13}$$

$$[\widetilde{X}, \alpha^v] = (\mathcal{L}_X \alpha)^v, \tag{2.14}$$

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]}. \tag{2.15}$$

Proof. This is easily verified in coordinates. For completeness, let us check (2.14), which is defined for $\alpha \in \mathcal{X}^*(E)$ and $X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$, meaning that we have to check this for $X \in \mathcal{X}_V(E)$ and $Y \in \mathcal{X}_t(E)$. For $X \in \mathcal{X}_V(E)$, we get

$$[\widetilde{X}, \alpha^v] = \left(X^j \frac{\partial \alpha_i}{\partial q^j} + \alpha_j \frac{\partial X^j}{\partial q^i} \right) \frac{\partial}{\partial p_i}$$

and

$$\mathcal{L}_X \alpha = \left(X^j \frac{\partial \alpha_i}{\partial q^j} + \alpha_j \frac{\partial X^j}{\partial q^i} \right) dq^i + \alpha_i \frac{\partial X^i}{\partial t} dt,$$

it immediately follows that (2.14) is fulfilled for all $X \in \mathcal{X}_V(E)$. On the other hand, for all $Y \in \mathcal{X}_t(E)$ we obtain

$$[\widetilde{Y}, \alpha^v] = \left(\frac{\partial \alpha_i}{\partial t} + Y^j \frac{\partial \alpha_i}{\partial q^j} + \alpha_j \frac{\partial Y^j}{\partial q^i} \right) \frac{\partial}{\partial p_i}$$

and

$$\mathcal{L}_Y \alpha = \left(\frac{\partial \alpha_i}{\partial t} + Y^j \frac{\partial \alpha_i}{\partial q^j} + \alpha_j \frac{\partial Y^j}{\partial q^i} \right) dq^i + \left(\alpha_i \frac{\partial Y^i}{\partial t} + Y(\alpha_0) \right) dt,$$

so also $Y \in \mathcal{X}_t(E)$ satisfies (2.14). □

Also a natural basis of 1-forms on $J^1\tau^*$, which is provided by pull backs of 1-forms on E, complemented by 1-forms of the type dF_X, with F_X as introduced in Definition 2.14, will be used later on. Therefore, it is interesting to see what is the result of the action of the local basis of vector fields on $J^1\tau^*$ on this local basis of 1-forms on $J^1\tau^*$.

Lemma 2.18. *Let $\alpha, \beta \in \mathcal{X}^*(E)$, $X \in \mathcal{X}_V(E)$ and $Y \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$, then*

$$\langle \beta^v, \pi^*\alpha \rangle = 0, \tag{2.16}$$

$$\langle \alpha^v, dF_X \rangle = \pi^*\langle X, \alpha \rangle \tag{2.17}$$

$$\langle \widetilde{Y}, \pi^*\alpha \rangle = \pi^*\langle Y, \alpha \rangle \tag{2.18}$$

$$\langle \widetilde{Y}, dF_X \rangle = F_{[Y,X]}. \tag{2.19}$$

Proof. The proof is a straightforward calculation in coordinates. For example the last relation for $Y \in \mathcal{X}_t(E)$ follows from

$$\begin{aligned}
\langle \widetilde{Y}, dF_X \rangle &= p_i \frac{\partial X^i}{\partial t} + p_i Y^j \frac{\partial X^i}{\partial q^j} - p_j X^i \frac{\partial Y^j}{\partial q^i} \\
&= p_i (Y(X^i) - X(Y^i)) \\
&= F_{[Y,X]}.
\end{aligned}$$

Remark that the Lie bracket of two vector fields in $\mathcal{X}_V(E) \cup \mathcal{X}_t(E)$ is always a vertical vector field on E, so $F_{[Y,X]}$ is well-defined. □

2.5 Lifting type (1,1) tensor fields to $J^1\tau^*$

In what follows, R will always denote a type (1,1) tensor field on E with the property $R(dt) = 0$. As such, $R_{\pi(m)}(m)$ is well defined for all $m \in J^1\tau^*$ because although m is not a covector at $\pi(m)$, it is an equivalence class of covectors $\bmod dt$.

In coordinates, the tensor fields under consideration have the form

$$R = R^i_j(t,q) \frac{\partial}{\partial q^i} \otimes dq^j + R^i_0(t,q) \frac{\partial}{\partial q^i} \otimes dt. \tag{2.20}$$

Note in particular that $R(\mathcal{X}(E)) \subset \mathcal{X}_V(E)$.

We define several lifting operations of such tensor fields: the result is, respectively, a vector field, a 1-form and a (1,1) tensor field on $J^1\tau^*$.

Definition 2.19. *The vertical lift R^v is a vector field on $J^1\tau^*$, determined by*

$$\begin{aligned}
R^v(\pi^* f) &= 0, \quad \forall f \in C^\infty(E) \\
R^v(F_X) &= F_{R(X)} \quad \forall X \in \mathcal{X}_V(E).
\end{aligned}$$

Note, $R(X)$ is a vertical vector field since $R(dt) = 0$ for all $X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$. In coordinates,

$$R^v = p_i R^i_j \frac{\partial}{\partial p_j}. \tag{2.21}$$

Having added a new type of vertical vector field to the picture, it is appropriate that we complement the Lie bracket properties listed in Lemma 2.17. For $\alpha \in \mathcal{X}^*(E)$ and $X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$, we have

$$[\alpha^v, R^v] = R(\alpha)^v, \tag{2.22}$$

$$[R_1^v, R_2^v] = (R_1R_2 - R_2R_1)^v, \tag{2.23}$$

$$[\widetilde{X}, R^v] = (\mathcal{L}_X R)^v. \tag{2.24}$$

Note that if $R(dt) = 0$, then for all $X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$, also $(\mathcal{L}_X R)(dt) = 0$, so the vertical lift of $\mathcal{L}_X R$ is well defined.

Definition 2.20. *The horizontal lift R^h is a 1-form on $J^1\tau^*$, which is defined pointwise by*

$$\langle X_m, R^h_m\rangle = \langle T\pi(X_m), R_{\pi(m)}(m)\rangle \quad \text{for all } m \in J^1\tau^*,\ X \in \mathcal{X}(J^1\tau^*).$$

R^h is a semi-basic 1-form on $J^1\tau^*$, which in coordinates reads

$$R^h = p_i R^i_j dq^j + p_i R^i_0 dt. \tag{2.25}$$

It immediately follows that

$$R^h(\widetilde{X}) = F_{R(X)}, \qquad \forall X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E). \tag{2.26}$$

To define the complete lift of a (1,1) tensor field to the cotangent bundle (see Definition 2.6), one normally makes use of the canonical symplectic form, but we don't have a symplectic structure on $J^1\tau^*$. Therefore, we look at the relations which were established as properties in the cotangent case (see Theorem 2.12) as a source of inspiration to come to an alternative definition here. The following lemma will be useful for that purpose.

Lemma 2.21. *If f is an arbitrary function on E, we have*

$$(f\alpha)^v = f\,\alpha^v, \quad \alpha \in \mathcal{X}^*(E)$$

$$(fR)^v = f\,R^v, \quad R \text{ (1,1) tensor field on } E$$

$$\widetilde{fX} = f\,\widetilde{X} - F_X\,(df)^v, \quad X \in \mathcal{X}_V(E)$$

$$\mathcal{L}_{fX}R = f\,\mathcal{L}_X R - X \otimes R(df) + R(X) \otimes df.$$

Proof. For completeness: the factor f on the right-hand side of the first three relations should actually be $\pi^*(f)$, but we try to avoid an overload of notations here and in what follows. The expressions can be checked, for example in coordinates. The first one follows easily from

$$(f\alpha)^v = f\alpha_i \frac{\partial}{\partial p_i} = f\,\alpha^v.$$

Likewise

$$(fR)^v = fp_i R^i_j \frac{\partial}{\partial p_j} = f\,R^v.$$

For the third one we have

$$\begin{aligned}
\widetilde{fX} &= fX^i\frac{\partial}{\partial q^i} - p_i\frac{\partial fX^i}{\partial q^j}\frac{\partial}{\partial p_j}\\
&= f\left(X^i\frac{\partial}{\partial q^i} - p_i\frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial p_j}\right) - p_iX^i\frac{\partial f}{\partial q^j}\frac{\partial}{\partial p_j}\\
&= f\,\widetilde{X} - F_X\,(df)^v.
\end{aligned}$$

At last,

$$\begin{aligned}
\mathcal{L}_{fX}R = {}& fX(R^i_j)\frac{\partial}{\partial q^i}\otimes dq^j + fR^i_j\left(\mathcal{L}_X\frac{\partial}{\partial q^i}\right)\otimes dq^j + fR^i_j\frac{\partial}{\partial q^i}\otimes dX^j\\
&- R^i_j\frac{\partial f}{\partial q^i}X\otimes dq^j + R^i_jX^j\frac{\partial}{\partial q^i}\otimes df\\
={}& f\,\mathcal{L}_XR - X\otimes R(df) + R(X)\otimes df.
\end{aligned}$$

□

Theorem 2.22. *Given a type (1,1) tensor field R on E with the property $R(dt) = 0$, there is a unique type (1,1) tensor field $\widetilde{R}$ on $J^1\tau^*$, which has the properties*

$$\widetilde{R}(\alpha^v) = R(\alpha)^v, \quad \forall\,\alpha \in \mathcal{X}^*(E) \tag{2.27}$$

$$\widetilde{R}(\widetilde{X}) = \widetilde{R(X)} + (\mathcal{L}_XR)^v, \quad \forall\,X \in \mathcal{X}_V(E)\cup\mathcal{X}_t(E). \tag{2.28}$$

$\widetilde{R}$ is called the complete lift of R to $J^1\tau^*$.

Proof. Note first that for both types of vector fields in (2.28), we have that $R(X)$ is vertical, so that $\widetilde{R(X)}$ makes sense. As indicated before, the set of vector fields of the form α^v and $\widetilde{X}$ considered in the above relations constitute a local basis for the vector fields on $J^1\tau^*$. To be specific, we need n independent $\alpha \in \mathcal{X}^*(E)$, n independent $X \in \mathcal{X}_V(E)$ and one $X \in \mathcal{X}_t(E)$ to generate such a basis. Imposing linearity over the module $C^\infty(J^1\tau^*)$ then further fixes $\widetilde{R}$ for all vector fields in $\mathcal{X}(J^1\tau^*)$. But for this approach to be consistent, we need to verify that our construction does not depend on the specific choice of independent 1-forms α and vector fields X on E. Since every other selection of independent 1-forms α and vector fields X will arise from a linear combination over $C^\infty(E)$ of the original, the point is to check that the defining relations behave properly under f-linear changes of $\alpha \in \mathcal{X}^*(E)$ and $X \in \mathcal{X}_V(E)$, with $f \in C^\infty(E)$. Using the results of the preceding lemma, we have that

$$\widetilde{R}((f\alpha)^v) = \widetilde{R}(f\alpha^v) = f\,\widetilde{R}(\alpha^v) = f\,(R(\alpha))^v = (R(f\alpha))^v.$$

On the other hand we have

$$\begin{aligned}\widetilde{R}(\widetilde{fX}) &= \widetilde{R}\big(f\widetilde{X} - F_X(df)^v\big) = f\widetilde{R}(\widetilde{X}) - F_X\widetilde{R}((df)^v)\\ &= f\left(\widetilde{R(X)} + (\mathcal{L}_X R)^v\right) - F_X(R(df))^v.\end{aligned}$$

This can be seen to match the sum of the following two expressions:

$$\widetilde{R(fX)} = \widetilde{fR(X)} = f\,\widetilde{R(X)} - F_{R(X)}(df)^v,$$

and

$$(\mathcal{L}_{fX}R)^v = f(\mathcal{L}_X R)^v - F_X(R(df))^v + F_{R(X)}(df)^v,$$

where we have used also the general property $(Y \otimes \beta)^v = F_Y\,\beta^v$. □

In coordinates, the complete lift of the tensor field R in (2.20) reads,

$$\begin{aligned}\widetilde{R} = R^i_j&\left(\frac{\partial}{\partial q^i} \otimes dq^j + \frac{\partial}{\partial p_j} \otimes dp_i\right) + R^i_0\frac{\partial}{\partial q^i} \otimes dt\\ &+ p_i\left(\frac{\partial R^i_j}{\partial q^k} - \frac{\partial R^i_k}{\partial q^j}\right)\frac{\partial}{\partial p_j} \otimes dq^k + p_i\left(\frac{\partial R^i_k}{\partial t} - \frac{\partial R^i_0}{\partial q^k}\right)\frac{\partial}{\partial p_k} \otimes dt. \end{aligned} \tag{2.29}$$

At this point, it is of some interest to compare $\widetilde{R}$ with the complete lift of R to the cotangent bundle T^*E (see Definition 2.11). Let us call the complete lift to T^*E, $\widetilde{R}_{T^*}$ to make a clear distinction with the complete lift to $J^1\tau^*$. In coordinates

(t, q^i, p_0, p_i) on T^*E, $\widetilde{R}_{T^*}$ is given by

$$\begin{aligned}\widetilde{R}_{T^*} = R^i_j \left(\frac{\partial}{\partial q^i} \otimes dq^j + \frac{\partial}{\partial p_j} \otimes dp_i\right) + R^i_0 \left(\frac{\partial}{\partial q^i} \otimes dt + \frac{\partial}{\partial p_0} \otimes dp_i\right) \\ + p_i \left(\frac{\partial R^i_j}{\partial q^k} - \frac{\partial R^i_k}{\partial q^j}\right) \frac{\partial}{\partial p_j} \otimes dq^k + p_i \left(\frac{\partial R^i_k}{\partial t} - \frac{\partial R^i_0}{\partial q^k}\right) \frac{\partial}{\partial p_k} \otimes dt \\ + p_i \left(\frac{\partial R^i_0}{\partial q^k} - \frac{\partial R^i_k}{\partial t}\right) \frac{\partial}{\partial p_0} \otimes dq^k. \end{aligned} \tag{2.30}$$

Considering the projection $\rho : T^*E \to J^1\tau^*$, we define, next to ρ-related vector fields (2.4), also ρ-related (1,1) tensor fields.

Definition 2.23. *Type (1,1) tensor fields U on T^*E and V on $J^1\tau^*$ are said to be ρ-related, if for all ρ-related pairs of vector fields $(Y, Z) \in \mathcal{X}(T^*E) \times \mathcal{X}(J^1\tau^*)$, we have that $U(Y)$ is ρ-related to $V(Z)$.*

The following is an alternative characterization.

Proposition 2.24. *U and V are ρ-related if for all $\sigma \in \mathcal{X}^*(J^1\tau^*)$ we have that $U(\rho^*\sigma) = \rho^*(V(\sigma))$.*

Proof. We obtain, if Y and Z are ρ-related vector fields,

$$\begin{aligned}\langle T\rho \circ U(Y), \sigma \circ \rho\rangle &= \langle U(Y), \rho^*\sigma\rangle \\ &= \langle Y, U(\rho^*\sigma)\rangle \\ &= \langle Y, \rho^*(V(\sigma))\rangle \\ &= \langle T\rho \circ Y, V(\sigma) \circ \rho\rangle \\ &= \langle Z \circ \rho, V(\sigma) \circ \rho\rangle \\ &= \langle V(Z) \circ \rho, \sigma \circ \rho\rangle,\end{aligned}$$

so U and V are ρ-related. □

Proposition 2.25. *$\widetilde{R}_{T^*}$ on T^*E and $\widetilde{R}$ on $J^1\tau^*$ are ρ-related.*

Proof. For all $\sigma = adt + b_i dq^i + c^i dp_i \in \mathcal{X}^*(J^1\tau^*)$,

$$\begin{aligned}\widetilde{R}(\sigma) = R^i_j b_i dq^j + R^i_j c^j dp_i + R^i_0 b_i dt + c^j p_i \left(\frac{\partial R^i_j}{\partial q^k} - \frac{\partial R^i_k}{\partial q^j}\right) dq^k \\ + c^k p_i \left(\frac{\partial R^i_0}{\partial q^k} - \frac{\partial R^i_k}{\partial t}\right) dt\end{aligned}$$

and

$$\widetilde{R}_{T^*}(\rho^*\sigma) = R^i_j b_i dq^j + R^i_j c^j dp_i + R^i_0 b_i dt + c^j p_i \left(\frac{\partial R^i_j}{\partial q^k} - \frac{\partial R^i_k}{\partial q^j} \right) dq^k$$
$$+ c^k p_i \left(\frac{\partial R^i_0}{\partial q^k} - \frac{\partial R^i_k}{\partial t} \right) dt,$$

by applying the result of the preceding proposition it then immediately follows that $\widetilde{R}_{T^*}$ and $\widetilde{R}$ are ρ-related. □

It is important for the sequel that we also pin down the adjoint action of $\widetilde{R}$ by using a natural local basis of 1-forms on $J^1\tau^*$. The adjoint action of $\widetilde{R}$ is determined by the following coordinate free relations, in which also the horizontal lift of a (1,1) tensor field exhibits its relevance.

Proposition 2.26. *The action on 1-forms of the complete lift $\widetilde{R}$ on $J^1\tau^*$ is fully determined by the relations*

$$\widetilde{R}(\pi^*\alpha) = \pi^*(R(\alpha)), \quad \forall\, \alpha \in \mathcal{X}^*(E) \tag{2.31}$$
$$\widetilde{R}(dF_X) = dF_{R(X)} - (\mathcal{L}_X R)^h, \quad \forall\, X \in \mathcal{X}_V(E). \tag{2.32}$$

Proof. Again, a straightforward coordinate calculation is sufficient to verify these relations. For $\alpha \in \mathcal{X}^*(E)$,

$$\widetilde{R}(\pi^*\alpha) = R^i_j \alpha_i dq^j + R^i_0 \alpha_0 dt = \pi^* R(\alpha).$$

On the other hand, $dF_X = X^i dp_i + p_i\ \partial X^i/\partial t\ dt + p_i\ \partial X^i/\partial q^j\ dq^j$, so

$$\widetilde{R}(dF_X) = R^i_j X^j dp_i + p_k R^i_j \frac{\partial X^k}{\partial q^i} dq^j + p_j R^i_0 \frac{\partial X^j}{\partial q^i} dt + p_k X^j \left(\frac{\partial R^k_j}{\partial q^l} - \frac{\partial R^k_l}{\partial q^j} \right) dq^l$$
$$+ p_i X^k \left(\frac{\partial R^i_k}{\partial t} - \frac{\partial R^i_0}{\partial q^k} \right) dt$$

while

$$dF_{R(X)} = R^i_j X^j dp_i + p_i \frac{\partial R^i_j}{\partial t} X^j dt + p_i R^i_j \frac{\partial X^j}{\partial t} dt + p_i \frac{\partial R^i_j}{\partial q^k} X^j dq^k + p_i R^i_j \frac{\partial X^j}{\partial q^k} dq^k$$

and

$$(\mathcal{L}_X R)^h = p_i \left(X^j \frac{\partial R^i_k}{\partial q^j} + R^i_j \frac{\partial X^j}{\partial q^k} - R^j_k \frac{\partial X^i}{\partial q^j} \right) dq^k$$
$$+ p_i \left(X^k \frac{\partial R^i_0}{\partial q^k} + R^i_j \frac{\partial X^j}{\partial t} - R^k_0 \frac{\partial X^i}{\partial q^k} \right) dt$$

from which the second relation follows. □

2.5.1 Further properties of $\widetilde{R}$

The main goal for this subsection is to compute the Nijenhuis torsion of the complete lift $\widetilde{R}$. For that we will need some auxiliary properties, for example information about the Lie derivatives of $\widetilde{R}$ with respect to vector fields of type α^v or $\widetilde{X}$. This prompts us to consider a lifting operation from a 2-form on E to a type (1,1) tensor field on $J^1\tau^*$.

Definition 2.27. *For $\omega \in \bigwedge^2(E)$ we define a type (1,1) tensor field ω^v on $J^1\tau^*$, called the vertical lift of ω, by the relations*

$$\omega^v(\alpha^v) = 0, \quad \forall\, \alpha \in \mathcal{X}^*(E) \tag{2.33}$$

$$\omega^v(\widetilde{X}) = (i_X\omega)^v, \quad \forall\, X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E). \tag{2.34}$$

The defining relations are obviously linear with respect to multiplication of α or $X \in \mathcal{X}_V(E)$ with a function on E, so that these relations will indeed produce a well-defined (1,1) tensor field. In coordinates, if

$$\omega = \tfrac{1}{2}\omega_{ij}(t,q)dq^i \wedge dq^j + \omega_{0i}(t,q)dt \wedge dq^i, \tag{2.35}$$

then

$$\omega^v = \omega_{ij}\frac{\partial}{\partial p_j} \otimes dq^i + \omega_{0j}\frac{\partial}{\partial p_j} \otimes dt. \tag{2.36}$$

Proposition 2.28. *The basic Lie derivatives of the complete lift $\widetilde{R}$ have the following expressions*

$$\mathcal{L}_{\widetilde{X}}\widetilde{R} = \widetilde{\mathcal{L}_X R}, \quad \forall\, X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E) \tag{2.37}$$

$$\mathcal{L}_{\alpha^v}\widetilde{R} = \big(-R\lrcorner\, d\alpha + d(R\alpha)\big)^v, \quad \forall\, \alpha \in \mathcal{X}^*(E). \tag{2.38}$$

Proof. Note in passing that for a 2-form ω, $R\lrcorner\,\omega$ is not the same as $i_R\omega$; by $R\lrcorner\,\omega$ we mean the 2-form defined by $R\lrcorner\,\omega(X,Y) = \omega(RX,Y)$. The proof is a simple matter of evaluating both sides of the above claims on a basis of vector fields on

$J^1\tau^*$, with $\beta \in \mathcal{X}^*(E)$, $Y \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$. The first expression follows from

$$\begin{aligned}
\mathcal{L}_{\widetilde{X}} \widetilde{R}(\beta^v) &= \mathcal{L}_{\widetilde{X}} (\widetilde{R}(\beta^v)) - \widetilde{R}([\widetilde{X}, \beta^v]) \\
&= \mathcal{L}_{\widetilde{X}} (R(\beta)^v) - \widetilde{R}((\mathcal{L}_X \beta)^v) \\
&= (\mathcal{L}_X (R\beta))^v - (R(\mathcal{L}_X \beta))^v \\
&= (\mathcal{L}_X R(\beta))^v \\
&= \widetilde{\mathcal{L}_X R}(\beta^v)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{\widetilde{X}} \widetilde{R}(\widetilde{Y}) &= \mathcal{L}_{\widetilde{X}} (\widetilde{R}(\widetilde{Y})) - \widetilde{R}([\widetilde{X}, \widetilde{Y}]) \\
&= \mathcal{L}_{\widetilde{X}} (\widetilde{R(Y)}) + \mathcal{L}_{\widetilde{X}} ((\mathcal{L}_Y R)^v) - \widetilde{R}(\widetilde{[X, Y]}) \\
&= \widetilde{\mathcal{L}_X (RY)} + \mathcal{L}_X (\mathcal{L}_Y R)^v - R\widetilde{([X, Y])} - (\mathcal{L}_{[X,Y]} R)^v \\
&= \widetilde{\mathcal{L}_X (RY)} + \mathcal{L}_Y (\mathcal{L}_X R)^v \\
&= \widetilde{\mathcal{L}_X R}(\widetilde{Y}).
\end{aligned}$$

For (2.38), we have

$$\mathcal{L}_{\alpha^v} \widetilde{R}(\beta^v) = \mathcal{L}_{\alpha^v} \big(\widetilde{R}(\beta^v)\big) - \widetilde{R}([\alpha^v, \beta^v]) = 0,$$

while

$$\begin{aligned}
\mathcal{L}_{\alpha^v} \widetilde{R}(\widetilde{Y}) &= \mathcal{L}_{\alpha^v} \big(\widetilde{R}(\widetilde{Y})\big) - \widetilde{R}([\alpha^v, \widetilde{Y}]) \\
&= \mathcal{L}_{\alpha^v} \big(\widetilde{RY} + (\mathcal{L}_Y R)^v\big) + \widetilde{R}\big((\mathcal{L}_Y \alpha)^v\big) \\
&= -(\mathcal{L}_{RY} \alpha)^v + \big(\mathcal{L}_Y R(\alpha)\big)^v + \big(R(\mathcal{L}_Y \alpha)\big)^v \\
&= \big(- \mathcal{L}_{RY} \alpha + \mathcal{L}_Y (R\alpha)\big)^v \\
&= \big(- i_{RY} d\alpha + i_Y d(R\alpha)\big)^v \\
&= \big(- i_Y (R \lrcorner\, d\alpha) + i_Y d(R\alpha)\big)^v,
\end{aligned} \tag{2.39}$$

from which the equality (2.38) now readily follows. In making these computations, we have made use of properties such as (2.13), (2.14), (2.15), (2.22) and (2.24) and of course the defining relations of Theorem 2.22. □

For the computation of the Nijenhuis torsion of $\widetilde{R}$ we will need the following results.

Lemma 2.29. *Let R and Q denote (1,1) tensor fields on E vanishing on dt, then we have*

$$\widetilde{R}(Q^v) = (Q \circ R)^v. \tag{2.40}$$

Proof. We evaluate the expression above on a natural basis of 1-forms on $J^1\tau^*$. Making use of Proposition 2.26 and Definition 2.19 we get for all $\alpha \in \mathcal{X}^*(E)$

$$\langle \widetilde{R}(Q^v), \pi^*\alpha \rangle = \langle Q^v, \widetilde{R}(\pi^*\alpha) \rangle = \langle Q^v, \pi^*(R(\alpha)) \rangle = 0,$$

and for all $X \in \mathcal{X}_V(E)$

$$\langle \widetilde{R}(Q^v), dF_X \rangle = \langle Q^v, \widetilde{R}(dF_X) \rangle = \langle Q^v, dF_{RX} - (\mathcal{L}_X R)^h \rangle = Q^v(F_{RX}) = Q^v(R^v(F_X)),$$

from which (2.40) follows. □

Proposition 2.30. *The action of $\widetilde{R}^2$ on vector fields on $J^1\tau^*$ is determined by*

$$\begin{aligned} \widetilde{R}^2(\alpha^v) &= \left(R^2(\alpha)\right)^v, \quad \alpha \in \mathcal{X}^*(E) \\ \widetilde{R}^2(\widetilde{X}) &= \widetilde{R^2(X)} + \left(\mathcal{L}_X R^2\right)^v + (i_X N_R)^v, \quad X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E). \end{aligned}$$

Proof. The proposition follows from the defining relations in Theorem 2.22 and the previous proposition:

$$\widetilde{R}^2(\alpha^v) = \widetilde{R}((R\alpha)^v) = \left(R^2(\alpha)\right)^v$$

and

$$\begin{aligned} \widetilde{R}^2(\widetilde{X}) &= \widetilde{R}(\widetilde{R(X)} + (\mathcal{L}_X R)^v) \\ &= \widetilde{R^2(X)} + (\mathcal{L}_{RX} R)^v + (\mathcal{L}_X R \circ R)^v \\ &= \widetilde{R^2(X)} + (\mathcal{L}_{RX} R)^v + \left(\mathcal{L}_X R^2 - R \circ \mathcal{L}_X R\right)^v \\ &= \widetilde{R^2(X)} + \left(\mathcal{L}_X R^2\right)^v + (i_X N_R)^v. \end{aligned}$$

Where we put for the Nijenhuis torsion (1.1) we also put $(i_X N_R)(Y) = N_R(X,Y)$ and recall then the property that for the action on vector fields

$$\begin{aligned} (i_X N_R)(Y) &= N_R(X,Y) \\ &= \mathcal{L}_{RX}(RY) + R^2(\mathcal{L}_X Y) - R(\mathcal{L}_{RX} Y) - R(\mathcal{L}_X(RY)) \\ &= (\mathcal{L}_{RX} R)(Y) - R((\mathcal{L}_X R)(Y)) \end{aligned}$$

so

$$i_X N_R = \mathcal{L}_{RX}R - R\circ\mathcal{L}_X R. \tag{2.41}$$

□

Proposition 2.31. *The Nijenhuis torsion of the complete lift $\widetilde{R}$ on $J^1\tau^*$ is determined by the following relations, where $\alpha,\beta\in\mathcal{X}^*(E)$ and $X,Y\in\mathcal{X}_V(E)\cup\mathcal{X}_t(E)$,*

$$\begin{aligned}
N_{\widetilde{R}}(\alpha^v,\beta^v) &= 0,\\
N_{\widetilde{R}}(\widetilde{X},\alpha^v) &= ((i_X N_R)(\alpha))^v,\\
N_{\widetilde{R}}(\widetilde{X},\widetilde{Y}) &= \widetilde{N_R(X,Y)} + (i_{[X,Y]}N_R)^v + \big(\mathcal{L}_Y(i_X N_R) - \mathcal{L}_X(i_Y N_R)\big)^v.
\end{aligned}$$

Proof. The proof is a matter of making use of the defining relations in Theorem 2.22 again, together with the results about $\widetilde{R}^2$ of the preceding proposition and a number of the bracket relations established before. Since the bracket of two vertical lifts of 1-forms is zero, the first relation is trivial. We give a sketch of the calculation for the other two relations. Starting from the definition of $N_{\widetilde{R}}$ and a first implementation of known properties, mainly from Theorem 2.22, we have for $N_{\widetilde{R}}(\widetilde{X},\alpha^v)$,

$$\begin{aligned}
N_{\widetilde{R}}(\widetilde{X},\alpha^v) = {}& [\widetilde{RX},(R\alpha)^v] + [(\mathcal{L}_X R)^v,(R\alpha)^v] + \widetilde{R}^2((\mathcal{L}_X\alpha)^v)\\
&- \widetilde{R}([\widetilde{RX},\alpha^v]) - \widetilde{R}([(\mathcal{L}_X R)^v,\alpha^v]) - \widetilde{R}([\widetilde{X},(R\alpha)^v]).
\end{aligned}$$

A subsequent use of known bracket relations reduces the right-hand side to the vertical lift of the following aggregation of terms:

$$\mathcal{L}_{RX}(R\alpha) - \mathcal{L}_X R(R\alpha) + R^2(\mathcal{L}_X\alpha) - R(\mathcal{L}_{RX}\alpha) + R(\mathcal{L}_X R(\alpha)) - R(\mathcal{L}_X(R\alpha)).$$

It is now a simple matter to simplify this expression further to

$$\mathcal{L}_{RX}R(\alpha) - \mathcal{L}_X R(R\alpha) = (i_X N_R)(\alpha),$$

where one has to keep in mind that the order of composition of (1,1) tensor fields changes, when passing from the action on vector fields to the adjoint action on 1-forms. Similarly, we get

$$\begin{aligned}
N_{\widetilde{R}}(\widetilde{X},\widetilde{Y}) = {}& [\widetilde{RX,RY}] + [\widetilde{RX},(\mathcal{L}_Y R)^v] + [(\mathcal{L}_X R)^v,\widetilde{RY}] + \widetilde{R}^2(\widetilde{[X,Y]})\\
&- \widetilde{R}(\widetilde{[RX,Y]}) - \widetilde{R}([(\mathcal{L}_X R)^v,\widetilde{Y}]) - \widetilde{R}(\widetilde{[X,RY]}) - \widetilde{R}([\widetilde{X},(\mathcal{L}_Y R)^v).
\end{aligned}$$

Using known bracket relations and Proposition 2.30, the right-hand side reduces to

$$\widetilde{N_R(X,Y)} + (i_{[X,Y]}N_R)^v + (Q_{(X,Y)})^v$$

where $Q_{(X,Y)}$ stands for

$$\begin{aligned} Q_{(X,Y)} &= \mathcal{L}_{RX}\mathcal{L}_Y R - \mathcal{L}_{RY}\mathcal{L}_X R + [\mathcal{L}_X R, \mathcal{L}_Y R] + \mathcal{L}_{[X,Y]}R^2 + \mathcal{L}_Y\mathcal{L}_X R \circ R \\ &\quad - \mathcal{L}_X\mathcal{L}_Y R \circ R - \mathcal{L}_{[RX,Y]}R - \mathcal{L}_{[X,RY]}R \\ &= \mathcal{L}_X R \circ \mathcal{L}_Y R - \mathcal{L}_Y R \circ \mathcal{L}_X R + R \circ \mathcal{L}_X\mathcal{L}_Y R - R \circ \mathcal{L}_Y\mathcal{L}_X R \\ &\quad + \mathcal{L}_Y\mathcal{L}_{RX}R - \mathcal{L}_X\mathcal{L}_{RY}R \\ &= \mathcal{L}_X\left(R \circ \mathcal{L}_Y R - \mathcal{L}_{RY}R\right) - \mathcal{L}_Y\left(R \circ \mathcal{L}_X R - \mathcal{L}_{RX}R\right) \\ &= \mathcal{L}_Y(i_X N_R) - \mathcal{L}_X(i_Y N_R). \end{aligned}$$

□

Theorem 2.32. *$N_{\widetilde{R}} = 0$ if and only if $N_R = 0$.*

Proof. From the results of the preceding proposition, it is clear that $N_R = 0$ implies $N_{\widetilde{R}} = 0$. Conversely, $N_{\widetilde{R}} = 0$ implies in particular that $((i_X N_R)(\alpha))^v = 0$ for all 1-forms α on E and $X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$, which is equivalent to saying that the 1-form $(i_X N_R)(\alpha) = 0 \mod dt$. In turn, this can be expressed as $\langle Y, (i_X N_R)(\alpha)\rangle = 0$ for all vertical Y. Looking at N_R again as vector-valued two-form, the conclusion is that

$$N_R(X,Y) = 0, \quad \forall\, Y \in \mathcal{X}_V(E),\ X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E). \tag{2.42}$$

If we now take $X \in \mathcal{X}_t(E)$ and $Y \in \mathcal{X}_t(E)$, we get

$$N_R(X,Y) = N_R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + N_R\left(X^i\frac{\partial}{\partial q^i}, \frac{\partial}{\partial t}\right) + N_R\left(X, Y^i\frac{\partial}{\partial q^i}\right).$$

The first term is zero because of the skew-symmetry of N_R and (2.42) guarantees that the last two terms are zero. In the end, the fact that N_R is known to be tensorial guarantees that $N_R(X,Y) = 0$ for all $X, Y \in \mathcal{X}(E)$. □

2.6 Poisson-Nijenhuis structure on $J^1\tau^*$

In Section 2.1 we introduced already the canonical Poisson structure Λ and corresponding Poisson map P on $J^1\tau^*$. It is of interest to characterize P also by its action on the local basis of 1-forms on $J^1\tau^*$ which we used in the previous section. For $\alpha \in \mathcal{X}^*(E)$ and $X \in \mathcal{X}_V(E)$, we have

$$P(\pi^*\alpha) = \alpha_i \frac{\partial}{\partial p_i} = \alpha^v, \tag{2.43}$$

$$P(dF_X) = p_j \frac{\partial X^j}{\partial q^i}\frac{\partial}{\partial p_i} - X^i \frac{\partial}{\partial q^i} = -\widetilde{X}, \tag{2.44}$$

and note for completeness that for the horizontal lift of a (1,1) tensor field R (cf. (2.25)),

$$P(R^h) = p_i R^i_j \frac{\partial}{\partial p_j} = R^v. \tag{2.45}$$

Now let R as before be a (1,1) tensor field on E with the property $R(dt) = 0$. We wish to investigate under what circumstances the complete lift $\widetilde{R}$ is a candidate to become a recursion operator for P or, expressed differently, for the couple $(P, \widetilde{R})$ to define a Poisson-Nijenhuis structure (see Definition 1.19) on $J^1\tau^*$. A preliminary condition to be satisfied is that $\widetilde{R}$ should commute with the Poisson map P. Now, using (2.31), (2.43) and (2.27), we get

$$P\widetilde{R}(\pi^*\alpha) = P(\pi^*R(\alpha)) = (R(\alpha))^v = \widetilde{R}(\alpha^v) = \widetilde{R}P(\pi^*\alpha).$$

Likewise, using (2.32), (2.44), (2.45) and (2.28), we have

$$P\widetilde{R}(dF_X) = P\big(dF_{R(X)} - (\mathcal{L}_X R)^h\big) = -\widetilde{R(X)} - (\mathcal{L}_X R)^v = -\widetilde{R}(\widetilde{X}) = \widetilde{R}P(dF_X).$$

This confirms the required commutation property, which is a condition also for the so-called Magri-Morosi concomitant $\mu_{\widetilde{R},P}$ (1.9) to be a tensor field of type $(2,1)$. The couple $(P, \widetilde{R})$ will define a Poisson-Nijenhuis structure if $\widetilde{R}$ has vanishing Nijenhuis torsion and if $\mu_{\widetilde{R},P} = 0$. We know already that $N_{\widetilde{R}}$ vanishes iff N_R vanishes. To check whether the Magri-Morosi concomitant vanishes, we further need the following list of properties of Lie derivatives of the 1-forms we obtained on $J^1\tau^*$ by lifting operations.

Lemma 2.33. *Let $\alpha, \beta \in \mathcal{X}^*(E)$, $X \in \mathcal{X}_V(E)$, $Y \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$, while R and Q denote (1,1) tensor fields on E vanishing on dt. Then,*

$$\mathcal{L}_{\beta^v}(\pi^*\alpha) = 0 \qquad \mathcal{L}_{Q^v}(\pi^*\alpha) = 0 \qquad \mathcal{L}_{\widetilde{Y}}(\pi^*\alpha) = \pi^*(\mathcal{L}_Y\alpha) \tag{2.46}$$

$$\mathcal{L}_{\beta^v} dF_X = \pi^* d i_X \beta \qquad \mathcal{L}_{Q^v} dF_X = dF_{QX} \qquad \mathcal{L}_{\widetilde{Y}} dF_X = dF_{[Y,X]} \tag{2.47}$$

$$\mathcal{L}_{\beta^v} R^h = \pi^* R(\beta) \qquad \mathcal{L}_{Q^v} R^h = (Q \circ R)^h \qquad \mathcal{L}_{\widetilde{Y}} R^h = (\mathcal{L}_Y R)^h. \tag{2.48}$$

Proof. The proof is a straightforward calculation which can be done either by evaluating both sides on the usual basis of vector fields and making use of Lemma 2.18 and Lie derivative properties obtained before, or perhaps more simply by a direct coordinate calculation. As an example, let us check the last equality. For $\alpha \in \mathcal{X}^*(E)$,

$$(\mathcal{L}_{\widetilde{Y}} R^h)(\alpha^v) = \mathcal{L}_{\widetilde{Y}}(R^h(\alpha^v)) - R^h(\mathcal{L}_{\widetilde{Y}}\alpha^v) = -R^h(\mathcal{L}_Y \alpha^v) = 0 = (\mathcal{L}_Y R)^h(\alpha^v)$$

and for $X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$,

$$\begin{aligned}
(\mathcal{L}_{\widetilde{Y}} R^h)(\widetilde{X}) &= \mathcal{L}_{\widetilde{Y}}(R^h(\widetilde{X})) - R^h(\mathcal{L}_{\widetilde{Y}}\widetilde{X}) \\
&= \mathcal{L}_{\widetilde{Y}} F_{R(X)} - R^h(\widetilde{[Y,X]}) \\
&= \langle \widetilde{Y}, dF_{R(X)} \rangle - F_{R([Y,X])} \\
&= F_{[Y,R(X)]} - F_{R([Y,X])} \\
&= F_{\mathcal{L}_Y R(X)} \\
&= (\mathcal{L}_Y R)^h(\widetilde{X}).
\end{aligned}$$

□

Proposition 2.34. *The Magri-Morosi concomitant $\mu_{\widetilde{R},P}$ vanishes identically.*

Proof. For an elegant proof, we simply let σ and Z in the defining relation (1.9) run over the set of 1-forms and vector fields on $J^1\tau^*$ which we have used all the time to generate a local basis. For $\sigma = \pi^*\alpha$ and $Z = \beta^v$, with $\alpha, \beta \in \mathcal{X}^*(E)$, we have for the right-hand side of (1.9)

$$(\mathcal{L}_{\alpha^v}\widetilde{R})(\beta^v) - P(\mathcal{L}_{\beta^v}(\pi^* R(\alpha))) + P(\mathcal{L}_{(R\beta)^v}\pi^*\alpha).$$

Using (2.46a), this reduces to

$$\mathcal{L}_{\alpha^v}(\widetilde{R}(\beta^v)) - \widetilde{R}(\mathcal{L}_{\alpha^v}\beta^v)$$

and these two terms vanish separately since the Lie bracket of two vertical vector fields is zero. For $\sigma = \pi^*\alpha$ and $Z = \widetilde{Y}$, the right-hand side of (1.9) reduces in the first place to

$$(\mathcal{L}_{\alpha^v}\widetilde{R})(\widetilde{Y}) - P(\mathcal{L}_{\widetilde{Y}}(\pi^* R(\alpha))) + P(\mathcal{L}_{\widetilde{RY}}\pi^*\alpha + \mathcal{L}_{\mathcal{L}_Y R^v}\pi^*\alpha).$$

This equals, making use of (4.86) and (2.46b, 2.46c),

$$-(\mathcal{L}_{RY}\alpha)^v + (\mathcal{L}_Y(R\alpha))^v - P(\pi^*\mathcal{L}_Y(R\alpha)) + P(\pi^*\mathcal{L}_{RY}\alpha),$$

and this is clearly zero in view of (2.43). Next, for $\sigma = dF_X$ and $Z = \beta^v$, using (2.44) in the first term, (2.32) in the second and (2.27) in the third, we get

$$-\big(\mathcal{L}_{\widetilde{X}}\widetilde{R}\big)(\beta^v) - P\big(\mathcal{L}_{\beta^v}(dF_{RX} - (\mathcal{L}_X R)^h)\big) + P\big(\mathcal{L}_{(R\beta)^v} dF_X\big).$$

We subsequently use (2.37) in the first term, (2.47a) and (2.48a) in the second and (2.47a) in the third again, and when we next evaluate the P-terms everything cancels out again. Finally, after a first evaluation, we get for $\mu_{\widetilde{R},P}(dF_X, \widetilde{Y})$:

$$-\big(\mathcal{L}_{\widetilde{X}}\widetilde{R}\big)(\widetilde{Y}) - P\big(\mathcal{L}_{\widetilde{Y}}(dF_{RX} - (\mathcal{L}_X R)^h)\big) + P\big(\mathcal{L}_{\widetilde{RY}} dF_X + \mathcal{L}_{(\mathcal{L}_Y R)^v} dF_X\big).$$

It is then a matter of using (2.47c), (2.48c) and (2.47b) to get

$$-(\widetilde{\mathcal{L}_X R})(\widetilde{Y}) - P(dF_{[Y,RX]}) + P((\mathcal{L}_Y\mathcal{L}_X R)^h) + P(dF_{[RY,X]}) + P(dF_{\mathcal{L}_Y R(X)})$$

using the properties (2.44) and (2.45) of P, we eventually come to an expression where everything cancels out again. □

The preceding properties lead to the following conclusion.

Theorem 2.35. *Let R be a (1,1) tensor field on $\tau : E \to \mathbb{R}$ with the property $R(dt) = 0$ and let $\widetilde{R}$ be its complete lift to $J^1\tau^*$. Denote by $P : \mathcal{X}^*(J^1\tau^*) \to \mathcal{X}(J^1\tau^*)$ the canonical Poisson map on $J^1\tau^*$. Then, $(P, \widetilde{R})$ is a Poisson-Nijenhuis structure on $J^1\tau^*$ if and only if $N_R = 0$, where N_R is the Nijenhuis torsion of R on E.*

Proof. We have shown that for any R satisfying $R(dt) = 0$, $P\widetilde{R} = \widetilde{R}P$ and that the Magri-Morosi concomitant $\mu_{\widetilde{R},P}$ vanishes identically. The only other requirement for having a Poisson-Nijenhuis structure then is that $N_{\widetilde{R}} = 0$. But Theorem 2.32 tells us that this is equivalent to $N_R = 0$. □

There is an important remark to be made here: the previous theorem also holds for T^*E.

Theorem 2.36. *Let R be a (1,1) tensor field on $\tau : E \to \mathbb{R}$ with the property $R(dt) = 0$ and let $\widetilde{R}_{T^*}$ be its complete lift to T^*E. Denote by $P_{T^*} : \mathcal{X}^*(T^*E) \to \mathcal{X}(T^*E)$ the Poisson map on T^*E corresponding with the canonical symplectic form $d\theta_E$. Then, $(P_{T^*}, \widetilde{R}_{T^*})$ is a Poisson-Nijenhuis structure on T^*E if and only if $N_R = 0$.*

Proof. Recall that a property of $\widetilde{R}_{T^*}$ which immediately follows from the defining relation (2.6) is its symmetry with respect to $d\theta_E$, meaning that

$$d\theta_E(\widetilde{R}_{T^*}(U), V) = d\theta_E(U, \widetilde{R}_{T^*}(V)), \qquad \forall U, V \in \mathcal{X}(T^*E). \tag{2.49}$$

Obviously, $\widetilde{R}_{T^*}$ then has a similar symmetry property with respect to the Poisson structure on T^*E which is the inverse of $d\theta_E$. The proof for the vanishing of the Magri-Morosi concomitant $\mu_{\widetilde{R}_{T^*}, P_{T^*}}$ can be found in [18]. The last condition is the vanishing of the Nijenhuis torsion, so the statement then immediately follows from Corollary 2.13. □

2.6.1 Darboux-Nijenhuis coordinates

In this subsection we will give a detailed explanation of the way in which Darboux-Nijenhuis coordinates arise for the case of our $(P, \widetilde{R})$ structure on $J^1\tau^*$.

The basic assumptions are that the tensor field R on E is *algebraically diagonalizable*, meaning that at each point $e \in E$, the endomorphism R_e of T_eE is diagonalizable, and that *the eigenvalues are distinct*. Observe then that the property $R(dt) = 0$ immediately says that dt is an eigenform corresponding to the eigenvalue $\lambda_0 = 0$, so that the remaining eigenvalues λ_i (which generally will be functions of the coordinates (t, q^i) of e) are non-zero by assumption. Obviously, R is degenerate, but our assumption implies that it has rank n. We write (R^α_β) now for the matrix representation of R (as linear map on vectors), with Greek indices running from 0 to n, and α in the role of row index. So, with R as in (2.20), the first row of the matrix has only zeros, the R^i_0 constitute the remaining elements of the first column, and the $n \times n$ matrix (R^i_j) is nonsingular:

$$(R^\alpha_\beta) = \begin{pmatrix} 0 & 0 \\ R^i_0 & R^i_j \end{pmatrix}.$$

The main point about the extra assumption of diagonalizability is that, in the context of a Poisson-Nijenhuis structure, one can do better than merely algebraically diagonalize. Indeed, it is known that one can perform a diagonalization in coordinates which will at the same time produce Darboux coordinates for the Poisson tensor (see Definition 1.20). Since the Poisson tensor Λ already takes its canonical form (2.2) in natural bundle coordinates (t, q, p) on $J^1\tau^*$, what we are after is a canonical coordinate transformation which does not destroy this canonical form and achieves the diagonalization of $\widetilde{R}$ in coordinates. Hence, the induced transformation of a time-dependent coordinate transformation on E which diagonalizes R in coordinates is a possible transformation to Darboux-Nijenhuis coordinates for $(P, \widetilde{R})$. From the fundamental paper of Frölicher and Nijenhuis [29], we learn that diagonalizability in coordinates requires vanishing of the so-called Haantjes tensor. But a direct application of this theory in our case, where the manifold E has coordinates $x^\alpha = (t, q^i)$ say, will merely guarantee the existence of new coordinates $y^\beta = y^\beta(x^\alpha)$ which do the job. This is a supplementary reason for going through the procedure in some detail here, because we need a transformation which preserves the fibred structure of E, i.e. the y^β should be of the form $(t, Q^i(t,q))$.

Let $X_{(\alpha)}$ be a local basis for $\mathcal{X}(E)$ consisting of eigenvector fields of R with corresponding eigenvalues $\lambda_{(\alpha)}$ (the extra brackets used for the index are meant to indicate that there are no summations over repeated indices in what follows). Then,

$$\begin{aligned} N_R(X_{(\alpha)}, X_{(\beta)}) = (R - \lambda_{(\alpha)})(R - \lambda_{(\beta)})([X_{(\alpha)}, X_{(\beta)}]) \\ + (\lambda_{(\alpha)} - \lambda_{(\beta)})\big(X_{(\alpha)}(\lambda_{(\beta)})\, X_{(\beta)} + X_{(\beta)}(\lambda_{(\alpha)})\, X_{(\alpha)}\big). \end{aligned}$$

Following [29], we next look at the Haantjes tensor, defined by

$$\mathcal{H}_R(X, Y) := R^2 N_R(X, Y) + N_R(RX, RY) - RN_R(RX, Y) - RN_R(X, RY),$$

and easily obtain that

$$\begin{aligned} \mathcal{H}_R(X_{(\alpha)}, X_{(\beta)}) &= (R - \lambda_{(\alpha)})(R - \lambda_{(\beta)}) N_R(X_{(\alpha)}, X_{(\beta)}) \\ &= (R - \lambda_{(\alpha)})^2 (R - \lambda_{(\beta)})^2 ([X_{(\alpha)}, X_{(\beta)}]). \end{aligned}$$

Since $N_R = 0$, also $\mathcal{H}_R = 0$ and this is the necessary and sufficient condition for a diagonalizable R to be diagonalizable in coordinates. The distribution spanned by $X_{(\alpha)}$, denoted by $\mathcal{D}_\alpha$, is integrable since it is by assumption 1-dimensional. In our case $\mathcal{H}_R = 0$ implies that $[X_{(\alpha)}, X_{(\beta)}] \in \mathrm{sp}\{X_{(\alpha)}, X_{(\beta)}\}$ for all α, β. So every sum of $\mathcal{D}_\alpha$'s is integrable and this implies that all $\mathcal{D}_\alpha$ are simultaneously integrable. In other words, there exist new coordinates y^α such that in those coordinates, $\mathcal{D}_\alpha = \mathrm{sp}\{\partial/\partial y^\alpha\}$. Our extra concern now is that such new coordinates should be of the form indicated above. To see that this is possible, it suffices to look at the dual picture of eigenforms. Let $\mathcal{D}_\alpha^\perp$ denote the annihilator of $\mathcal{D}_\alpha$ and put $\mathcal{D}_\alpha^* = \cap_{\beta\neq\alpha} \mathcal{D}_\beta^\perp$. Then by construction $\langle X_{(\beta)}, \rho_{(\alpha)}\rangle = 0$ for $\forall \beta \neq \alpha$ and $\rho_{(\alpha)} \in \mathcal{D}_\alpha^*$, while

$$\langle X_{(\alpha)}, R(\rho_{(\alpha)})\rangle = \langle R(X_{(\alpha)}), \rho_{(\alpha)}\rangle = \lambda_{(\alpha)} \langle X_{(\alpha)}, \rho_{(\alpha)}\rangle.$$

Hence, $\rho_{(\alpha)}$ is an eigenform of R corresponding to the eigenvalue $\lambda_{(\alpha)}$. Therefore, in the coordinates y^α simultaneously adapted to all $\mathcal{D}_\alpha$, $\rho_{(\alpha)}$ is in the module generated by dy^α. So, in the dual picture, the coordinate transformation $(t,q^i) \to y^\alpha$ has the task of producing eigenforms of the form dy^α. But we know already that dt is an eigenform (with eigenvalue zero), hence we can simply take $y^0 = t$, meaning that R will indeed be diagonalized by a transformation of the form $(t,q) \to (t, Q(t,q))$. In the new variables, the eigenvectors can be taken to be coordinate fields, so that they commute and the Nijenhuis tensor expression reduces to

$$N_R(X_{(\alpha)}, X_{(\beta)}) = (\lambda_{(\alpha)} - \lambda_{(\beta)})\big(X_{(\alpha)}(\lambda_{(\beta)})\, X_{(\beta)} + X_{(\beta)}(\lambda_{(\alpha)})\, X_{(\alpha)}\big).$$

It follows that the stronger property $N_R = 0$ now implies that $X_{(\alpha)}(\lambda_{(\beta)}) = 0$ for $\alpha \neq \beta$. The conclusion is that each eigenvalue $\lambda_{(\alpha)}$ is a function of the corresponding coordinate y^α only or thus that R is separable in coordinates. The final conclusion is that in the new coordinates (t,Q), R will take the form

$$R = \sum_{i=1}^{n} \lambda_{(i)}(Q^i)\frac{\partial}{\partial Q^i} \otimes dQ^i. \tag{2.50}$$

As said before, the nature of the coordinate transformation involved in this process ensures that the Poisson tensor Λ will still have the canonical form

$$\Lambda = \frac{\partial}{\partial Q^i} \wedge \frac{\partial}{\partial P_i},$$

so we have indeed obtained Darboux-Nijenhuis coordinates for the Poisson-Nijenhuis structure $(P, \widetilde{R})$. It is an easy matter to verify that the same point transformation on E will also induce a canonical transformation on T^*E that defines Darboux-Nijenhuis coordinates for $(P_{T^*}, \widetilde{R}_{T^*})$.

Let us, to conclude this chapter, summarize this important result in the following theorem.

Theorem 2.37. *Let E be a bundle over $\mathbb{R}$ of dimension $n+1$. Let R be a type (1,1) tensor field on E which has the property $R(dt) = 0$. Suppose that $N_R = 0$ and that R is algebraically diagonalizable with distinct eigenvalues. Then, there locally exists a coordinate transformation $(t,q) \to (t, Q(t,q))$ on E, which induces Darboux-Nijenhuis coordinates for both Poisson-Nijenhuis structures $(P_{T^*}, \widetilde{R}_{T^*})$ and $(P, \widetilde{R})$ on T^*E and $J^1\tau^*$, respectively.*

It follows that the coordinate expressions of $\widetilde{R}_{T^*}$ and $\widetilde{R}$ in Darboux-Nijenhuis coordinates formally are identical and read,

$$\widetilde{R}_{T^*} = \widetilde{R} = \sum_{i=1}^{n} \lambda_i(Q^i) \left(\frac{\partial}{\partial Q^i} \otimes dQ^i + \frac{\partial}{\partial P_i} \otimes dP_i \right). \tag{2.51}$$

CHAPTER

3

INTRINSIC FORMULATION OF FORBAT'S CONDITIONS

As we discussed in Chapter 1, Levi-Civita developed necessary and sufficient conditions for separability of the Hamilton-Jacobi equation for autonomous Hamiltonian systems (1.14). Much less known is that these conditions were generalized to the case of time-dependent systems by Forbat [30]. Forbat's conditions for separability of a time-dependent Hamiltonian $H(t, q, p)$ in the coordinates (t, q^i, p_i) read

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial^2 H}{\partial q^i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial q^i\,\partial p_j}\frac{\partial H}{\partial q^j}\right) = \frac{\partial H}{\partial q^i}\left(\frac{\partial^2 H}{\partial p_i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial p_i\,\partial p_j}\frac{\partial H}{\partial q^j}\right), \tag{3.1}$$

$$\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial q^i\,\partial t} = \frac{\partial H}{\partial q^i}\frac{\partial^2 H}{\partial p_i\,\partial t}, \tag{3.2}$$

where there is no summation over repeated indices and $i, j = 1, \ldots, n$, $i \neq j$. A study of Hamiltonians of mechanical type which satisfy Forbat's conditions was conducted in [12]. A weak point about these conditions is that they can merely test whether the Hamilton-Jacobi equation is separable in the given coordinates: one has to be lucky to have chosen separation coordinates already for the test to give positive results. In this chapter, our purpose is to develop an intrinsic formulation of Forbat's conditions, i.e. to obtain a test for the existence of separation coordinates which in principle can be carried out in any given coordinate chart and should then provide information about the way separation coordinates can be constructed.

3.1 Time-dependent Hamiltonian systems

As we argued already in Chapter 2, we choose $J^1\tau^*$, the dual of the first jet bundle, $J^1\tau$, of a bundle $\tau : E \to \mathbb{R}$, as the geometric framework for our study of time-dependent Hamiltonian systems. So briefly, let $\tau : E \to \mathbb{R}$ be a bundle with $\dim E = n+1$ and coordinates denoted by (t, q^i) and let $J^1\tau^*$ be the dual of $J^1\tau$. There are natural projections, say $\rho : T^*E \to J^1\tau^*$ and $\pi : J^1\tau^* \to E$. A Hamiltonian is defined as a section h of the line bundle $\rho : T^*E \to J^1\tau^*$. Locally, h determines a function H on $J^1\tau^*$, according to

$$h : J^1\tau^* \to T^*E : (t, q, p) \mapsto (t, q, p_0 = -H(t, q, p), p).$$

In fact, $h(J^1\tau^*)$ is a submanifold of T^*E which is locally determined by the equation $\widetilde{H} := p_0 + H(t, q, p) = 0$. Moreover, if we restrict ρ to the image of h, ρ and h are each others inverse. If $\omega_E = d\theta_E$ denotes the canonical symplectic form on T^*E, we have that locally $h^*\omega_E = dp_i \wedge dq^i - dH \wedge dt$. The associated Hamiltonian vector field on $J^1\tau^*$, denoted by X_h, is defined by

$$i_{X_h} h^*\omega_E = 0 \quad \text{and} \quad \langle X_h, dt \rangle = 1,$$

and is locally of the form (here with the usual summation convention)

$$X_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{3.3}$$

In principle, one may hope to develop an intrinsic model for the separability issue directly on $J^1\tau^*$, the manifold where X_h lives. However, one should be cautious: it is well known that the Hamiltonian function H on $J^1\tau^*$ picks up extra terms under a time-dependent canonical transformation. Explicitly, consider a time-dependent coordinate transformation on E, $(t, q) \to (t, Q(t, q))$, and let $(t, q, p_0, p) \to (t, Q(t, q), P_0(t, q, p_0, p), P(t, q, p))$ be the induced time-dependent canonical transformation on T^*E. In other words, we have that

$$P_0 = p_0 + p_i \frac{\partial q^i}{\partial t}(t, Q(t, q)) \quad \text{and} \quad P_i(t, q, p) = p_l \frac{\partial q^l}{\partial Q^i}(t, Q(t, q)).$$

Expressed in the new coordinates the function $\widetilde{H}$ reads

$$\widetilde{H} = P_0 - p_i(t,Q,P)\frac{\partial q^i}{\partial t}(t,Q) + H(t,q(t,Q),p(t,Q,P))$$

so the Hamiltonian $K(t,Q,P)$ in the new coordinates will be given by

$$K(t,Q,P) = H(t,q(t,Q),p(t,Q,P)) - p_i(t,Q,P)\frac{\partial q^i}{\partial t}(t,Q). \tag{3.4}$$

Thus in a way, there is no life for X_h without the presence of T^*E, which therefore has to remain in the picture.

3.2 Distributions associated to functions on T^*E

Before starting, we should say that we owe a great deal of the inspiration for the distributions under consideration to a private meeting of Willy Sarlet with Franco Magri back in 2001. To the best of our knowledge, Magri's ideas were never published, but they were the source of inspiration also for some of the results reported in [21] and [50], both for autonomous Hamiltonian systems.

As in the previous chapter, let R be a (1,1) tensor field on E which vanishes on dt and suppose R is *algebraically diagonalizable* with *distinct eigenvalues*. We denote the complete lift of R to T^*E again by $\widetilde{R}_{T^*}$ and likewise the complete lift of R to $J^1\tau^*$ will be denoted by $\widetilde{R}$. From the coordinate expressions (2.30) and (2.29), it is clear that the coefficient matrices of $\widetilde{R}_{T^*}$ and $\widetilde{R}$ have a block matrix structure. In particular, they have a $n \times n$ zero block corresponding to the lack of terms of the form $\partial/\partial q^i \otimes dp_j$, and have twice the block matrix (R^i_j) along the diagonal. It then further readily follows that $\widetilde{R}_{T^*}$ has double eigenvalues $(0, \lambda_i)$ and $\widetilde{R}$ has a single eigenvalue 0 and double eigenvalues λ_i. As a result, both matrices have $\lambda \prod_{i=1}^n (\lambda - \lambda_i)$ as their minimal polynomial.

Now, let F be a function on T^*E and consider the distribution

$$\mathcal{D}_F = \mathrm{sp}\,\{dF, \widetilde{R}_{T^*}(dF), \widetilde{R}^2_{T^*}(dF), \ldots, \widetilde{R}^n_{T^*}(dF)\}^\circ$$

consisting of the set of vector fields annihilating the indicated 1-forms. The sequence of these 1-forms certainly breaks down at the power n of $\widetilde{R}_{T^*}$, because the minimal polynomial of $\widetilde{R}_{T^*}$ has degree $n+1$. Assume that F and R are such that the defining 1-forms are linearly independent (except for isolated points), so that $\mathcal{D}_F$ also has dimension $n+1$ and will be Lagrangian, provided it is isotropic or co-isotropic.

Lemma 3.1. *The (symplectic) orthogonal complement $\mathcal{D}_F^\perp$ of $\mathcal{D}_F$ is given by*

$$\mathcal{D}_F^\perp = \mathrm{sp}\,\{X_F, \widetilde{R}_{T^*}(X_F), \ldots, \widetilde{R}_{T^*}^n(X_F)\}.$$

Proof. For all $Y \in \mathcal{D}_F$ and $k = 0, \ldots, n$ we have, using the symmetry of $\widetilde{R}_{T^*}$ with respect to ω_E, that

$$\omega_E(\widetilde{R}_{T^*}^k(X_F), Y) = \omega_E(X_F, \widetilde{R}_{T^*}^k(Y)) = -dF(\widetilde{R}_{T^*}^k(Y)) = -\widetilde{R}_{T^*}^k(dF)(Y) = 0,$$

which shows that $\widetilde{R}_{T^*}^k(X_F)$ belongs to $\mathcal{D}_F^\perp$ for $k = 0, \ldots, n$. Moreover, in open domains where the defining 1-forms of $\mathcal{D}_F$ are linearly independent, the same is true for the vector fields $\widetilde{R}_{T^*}^k(X_F)$. Indeed, if $\sum_{k=0}^n a_k \widetilde{R}_{T^*}^k(X_F) = 0$, we have for all $Z \in \mathcal{X}(T^*E)$,

$$0 = \omega_E\Big(\sum_{k=0}^n a_k \widetilde{R}_{T^*}^k(X_F), Z\Big) = -\sum_{k=0}^n a_k \langle \widetilde{R}_{T^*}^k(Z), dF\rangle = -\Big\langle Z, \sum_{k=0}^n a_k \widetilde{R}_{T^*}^k(dF)\Big\rangle$$

which implies that all functions a_k must be zero. By dimension, therefore, the $\widetilde{R}_{T^*}^k(X_F)$ span $\mathcal{D}_F^\perp$. □

Lemma 3.2. *The distribution $\mathcal{D}_F$ is Lagrangian, i.e. $\mathcal{D}_F = \mathcal{D}_F^\perp$.*

Proof. We will show that $\mathcal{D}_F$ is co-isotropic, i.e. $\mathcal{D}_F^\perp \subseteq \mathcal{D}_F$. For that purpose, consider

$$\Big\langle \widetilde{R}_{T^*}^l(X_F), \widetilde{R}_{T^*}^k(dF)\Big\rangle = \Big\langle \widetilde{R}_{T^*}^{k+l}(X_F), dF\Big\rangle = -\omega_E\big(X_F, \widetilde{R}_{T^*}^{k+l}(X_F)\big).$$

Again, by the symmetry of $\widetilde{R}_{T^*}$ with respect to ω_E, we have

$$\omega_E\big(X_F, \widetilde{R}_{T^*}^{k+l}(X_F)\big) = \omega_E\big(\widetilde{R}_{T^*}^{k+l}(X_F), X_F\big),$$

but then the skew-symmetry of ω_E implies that this is identically zero. Since this is valid for all l and k, we conclude from the first line that $\mathcal{D}_F^\perp \subseteq \mathcal{D}_F$. The dimension then implies that we have equality and thus a Lagrangian distribution. □

In what follows, we will denote the distribution simply by $\mathcal{D}_F$ even when we appeal to the defining relation of $\mathcal{D}_F^\perp$. Note further that it follows from both defining relations and the degree of the minimal polynomial of $\widetilde{R}_{T^*}$ that $\widetilde{R}_{T^*}(\mathcal{D}_F) \subset \mathcal{D}_F$.

Naturally, we are interested in the case that $\mathcal{D}_F$ is Frobenius integrable.

Theorem 3.3. *Let R be a $(1,1)$ tensor field on E with the property $R(dt) = 0$, which is algebraically diagonalizable with distinct eigenvalues and has vanishing Nijenhuis torsion, and F a function on T^*E for which the defining 1-forms of the distribution $\mathcal{D}_F$ are linearly independent. Then $\mathcal{D}_F$ is integrable if and only if $dd_{\widetilde{R}_{T^*}}F\big|_{\mathcal{D}_F} = 0$.*

Proof. Looking at the defining co-distribution of $\mathcal{D}_F$ and putting $\alpha_i = d_{\widetilde{R}^i_{T^*}}F$ for shorthand, Frobenius theorem implies that $\mathcal{D}_F$ is integrable if and only if $d\alpha_i = \sum_{l=0}^n \theta_i^l \wedge \alpha_l$ for some 1-forms θ_i^l. By extending the α_i to a local basis for $\mathcal{X}^*(T^*E)$, it is easy to see that this is further equivalent to $d\alpha_i\big|_{\mathcal{D}_F} = 0$ for all i. Hence, if $\mathcal{D}_F$ is integrable, we have in particular that $d\alpha_1\big|_{\mathcal{D}_F} = dd_{\widetilde{R}_{T^*}}F\big|_{\mathcal{D}_F} = 0$.

Conversely, assuming $dd_{\widetilde{R}_{T^*}}F\big|_{\mathcal{D}_F} = 0$, we first observe that $d\alpha_0\big|_{\mathcal{D}_F} = 0$ since $d\alpha_0 = d(dF) = 0$, and also

$$d_{\widetilde{R}_{T^*}}\alpha_0\big|_{\mathcal{D}_F} = -di_{\widetilde{R}_{T^*}}\alpha_0\big|_{\mathcal{D}_F} = -dd_{\widetilde{R}_{T^*}}F\big|_{\mathcal{D}_F} = 0.$$

We now proceed further by induction. First note that $\alpha_i = d_{\widetilde{R}^i_{T^*}}F = \widetilde{R}^i_{T^*}(dF) = \widetilde{R}_{T^*}(\alpha_{i-1})$. Assuming that $d\alpha_i\big|_{\mathcal{D}_F} = 0$ and $d_{\widetilde{R}_{T^*}}\alpha_i\big|_{\mathcal{D}_F} = 0$, we will show that the same properties hold for α_{i+1}. Firstly, for all $X, Y \in \mathcal{D}_F$, using (1.2) and the fact that $N_{\widetilde{R}_{T^*}} = 0$, we conclude that

$$d_{\widetilde{R}_{T^*}}\alpha_{i+1}(X,Y) = d_{\widetilde{R}_{T^*}}(\widetilde{R}_{T^*}\alpha_i)(X,Y) = d\alpha_i(\widetilde{R}_{T^*}X, \widetilde{R}_{T^*}Y) = 0,$$

since $\widetilde{R}_{T^*}(\mathcal{D}_F) \subset \mathcal{D}_F$. Secondly,

$$d\alpha_{i+1}(X,Y) = d(\widetilde{R}_{T^*}\alpha_i)(X,Y) = di_{\widetilde{R}_{T^*}}\alpha_i(X,Y) = i_{\widetilde{R}_{T^*}}d\alpha_i(X,Y) - d_{\widetilde{R}_{T^*}}\alpha_i(X,Y),$$

which reduces to the first term on the right by the induction hypothesis and then in fact, since

$$i_{\widetilde{R}_{T^*}}d\alpha_i(X,Y) = d\alpha_i(\widetilde{R}_{T^*}X, Y) + d\alpha_i(X, \widetilde{R}_{T^*}Y),$$

to zero in view of $\widetilde{R}_{T^*}(\mathcal{D}_F) \subset \mathcal{D}_F$ and the induction hypothesis again. The conclusion is that, in particular, $d\alpha_i\big|_{\mathcal{D}_F} = 0$ for all i and hence that $\mathcal{D}_F$ is integrable. $\square$

There is a direct link between the integrability of $\mathcal{D}_F$ and the classical Hamilton-Jacobi equation for the Hamiltonian F on T^*E as we will now demonstrate. First recall that a *regular point* of $\mathcal{D}_F$ is a point where the distribution is transversal to the fibers, i.e. $\mathcal{D}_F$ contains no vertical vector fields except the zero-vector field. A linear combination of the vector fields spanning $\mathcal{D}_F$ is a vertical vector field if and only if its projection onto E is zero. So, for the point (t, q, p_0, p) to be a regular point, the projection of the vector fields which span $\mathcal{D}_F$ should be linearly independent. If $\pi_E = \pi \circ \rho$ denotes the projection of T^*E onto E, we have at each point (t, q, p_0, p) of T^*E that

$$T\pi_E\left(\sum_{k=0}^{n} c_k \widetilde{R}^k_{T^*}(X_F)\right) = \sum_{k=0}^{n} c_k T\pi_E(\widetilde{R}^k_{T^*}(X_F))$$

and

$$\begin{aligned}
T\pi_E(X_F(t,q,p_0,p)) &= \frac{\partial F}{\partial p_0}\left.\frac{\partial}{\partial t}\right|_{(t,q)} + \frac{\partial F}{\partial p_i}\left.\frac{\partial}{\partial q^i}\right|_{(t,q)} \\
T\pi_E\big(\widetilde{R}_{T^*}(X_F)(t,q,p_0,p)\big) &= \left(R^i_j\frac{\partial F}{\partial p_j} + R^i_0\frac{\partial F}{\partial p_0}\right)\left.\frac{\partial}{\partial q^i}\right|_{(t,q)} \\
&\vdots \\
T\pi_E\big(\widetilde{R}^n_{T^*}(X_F)(t,q,p_0,p)\big) &= R^{(n-1)}{}^i_l\left(R^l_j\frac{\partial F}{\partial p_j} + R^l_0\frac{\partial F}{\partial p_0}\right)\left.\frac{\partial}{\partial q^i}\right|_{(t,q)}.
\end{aligned}$$

As R is algebraically diagonalizable, let X_l, $l = 0, \ldots, n$ be a local basis for $\mathcal{X}(E)$ consisting of eigenvector fields of R. Then if

$$T\pi_E(X_F(t,q,p_0,p)) = \sum_{l=0}^{n} a_l \left. X_l\right|_{(t,q)},$$

also

$$T\pi_E\big(\widetilde{R}^m_{T^*}(X_F)(t,q,p_0,p)\big) = \sum_{l=1}^{n} a_l\lambda_l^m \left. X_l\right|_{(t,q)}, \qquad m = 1, \ldots, n.$$

This implies that the projected vector fields are linearly independent if and only if $T\pi_E(X_F(t,q,p_0,p))$ is a linear combination (with non-zero coefficients) of all eigenvectors of the tensor R at the point $(t,q) = \pi_E(t,q,p_0,p)$. Equivalently, this means that $\partial F/\partial p_0 \neq 0$ and the vector with components $R^i_j(\partial F/\partial p_j) + R^i_0(\partial F/\partial p_0)$ is spanned by all eigenvectors of the matrix (R^i_j). Remark that in such a regular point, the vectors spanning $\mathcal{D}_F$ will be linearly independent if and only if their projections onto E are linearly independent.

If the Lagrangian distribution $\mathcal{D}_F$ is integrable it induces a foliation of T^*E into maximally connected integral submanifolds, the so-called leaves of the foliation, which in this case are Lagrangian submanifolds. By a result, proven e.g. in [62], integrability of $\mathcal{D}_F$ moreover implies that each point e of T^*E admits a coordinate neighbourhood U, with coordinates (x^I, α_I) for $I = 0, \ldots, n$, such that each leaf of the foliation intersects U at most in one connected slice, i.e. a submanifold characterized by fixed $\alpha = (\alpha_0, \ldots, \alpha_n)$. For each α we denote the corresponding slice by L_α which is then parametrized by the coordinates (x^I). Assuming the point e is a regular point of the distribution $\mathcal{D}_F$ it follows (possibly after shrinking the coordinate neighbourhood U, if need be) that all the slices L_α are transversal to the fibers and project diffeomorphically onto an open neighbourhood $\bar{U}_\alpha$ of $\pi_E(e) \in E$. Hence, we can take $x^I = q^I (= (t, q^1, \ldots, q^n))$, i.e. the coordinates (q^I) on E can also be regarded as coordinates parametrizing the slices of the foliation. Moreover, L_α being a Lagrangian submanifold, transversal to the fibres of T^*E, one can find a function $S_\alpha : \bar{U}_\alpha \to \mathbb{R}$ such that, again after possibly restricting $\bar{U}_\alpha$,

$$L_\alpha = dS_\alpha(\bar{U}_\alpha)\,.$$

We can then construct a function S on U as follows:

$$S : U \to \mathbb{R} \,:\, (q^I, \alpha_I) \mapsto S(q^I, \alpha_I) := S_\alpha(q^I)\,.$$

Defining $d_1 S$ by

$$d_1 S : U \to T^*E \,:\, (q^I, \alpha_I) \mapsto d_1 S(q^I, \alpha_I) = dS_\alpha(q^I) = \left(q^I, p_I = \frac{\partial S_\alpha}{\partial q^I}\right)\,,$$

and expressing that this is a (local) diffeomorphism onto its image, we have that its Jacobian matrix must be nonsingular: i.e.

$$\det \begin{pmatrix} \dfrac{\partial q^I}{\partial q^J} & \dfrac{\partial q^I}{\partial \alpha^J} \\ \dfrac{\partial p_I}{\partial q^J} & \dfrac{\partial p_I}{\partial \alpha^J} \end{pmatrix} \neq 0$$

which is equivalent to

$$\det \left(\frac{\partial^2 S}{\partial q^I \, \partial \alpha_J} \right) \neq 0.$$

Now, since $X_F \in \mathcal{D}_F$ by construction, F will be constant on every leaf of the foliation and, in particular,

$$F \circ d_1 S = \text{constant}\,.$$

This reads in coordinates

$$F\left(q^I, \frac{\partial S}{\partial q^I}\right) = \text{constant}\,,$$

which shows that S is (locally) a complete solution of the Hamilton-Jacobi equation for F. For an excellent and extensive account of the geometry of the Hamilton-Jacobi equation we refer to [44]. Observe, though, that in all references cited in this context, the base manifold E for the time-dependent case is taken to be a product manifold $\mathbb{R} \times Q$.

This brings us to the point that the above type of Hamilton-Jacobi equation is of course not exactly what we are interested in: it is in some sense Hamilton-Jacobi theory for an autonomous Hamiltonian where one of the coordinates q^I happens to be t. The case of interest is when F is of the form $F = \widetilde{H} := p_0 + H(t, q^i, p_i)$, and hence $\widetilde{H} = 0$ defines a section of $\rho : T^*E \to J^1\tau^*$ and a time-dependent Hamiltonian system on $J^1\tau^*$. The corresponding Hamilton-Jacobi equation then is of the form:

$$\frac{\partial S}{\partial t} + H\Big(t, q^i, \frac{\partial S}{\partial q^i}\Big) = 0. \tag{3.5}$$

For this case, we study the existence of a corresponding integrable distribution on $J^1\tau^*$ in the next section.

3.3 A corresponding distribution on $J^1\tau^*$ for given section $h : J^1\tau^* \to T^*E$

Consider a section $h : J^1\tau^* \to T^*E$, locally given by $h(t, q, p) = (t, q, -H(t, q, p), p)$ whose image is the set of points in T^*E for which $\widetilde{H} \equiv p_0 + H(t, q, p) = 0$. We have a corresponding Hamiltonian vector field $X_{\widetilde{H}}$ on T^*E, with coordinate expression,

$$X_{\widetilde{H}} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial t}\frac{\partial}{\partial p_0}, \tag{3.6}$$

and the Hamiltonian vector field X_h on $J^1\tau^*$ as defined in Section 3.1,

$$X_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

Clearly, $X_{\widetilde{H}}$ projects onto X_h, in other words $X_{\widetilde{H}}$ and X_h are ρ-related. We have already mentioned in Proposition 2.25 that also the tensor fields $\widetilde{R}_{T^*}$ and $\widetilde{R}$ are ρ-related. As a result, if we consider the distribution $\mathcal{D}_h$ on $J^1\tau^*$, defined by

$$\mathcal{D}_h = \mathrm{sp}\,\{X_h, \widetilde{R}(X_h), \widetilde{R}^2(X_h), \ldots, \widetilde{R}^n(X_h)\}, \tag{3.7}$$

it is clear that $\mathcal{D}_{\widetilde{H}}$ on T^*E and $\mathcal{D}_h$ on $J^1\tau^*$ are ρ-related. We wish to show now that, more importantly, they are also h-related. A vector field X on $J^1\tau^*$ is h-related to Y on T^*E if $Th \circ X = Y \circ h$ or equivalently $Y(F) \circ h = X(F \circ h)$ for all functions F on T^*E. It is easy to verify that this translates into the following relationship between the local coordinate expressions. If

$$X = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + U_i \frac{\partial}{\partial p_i},$$

then, since

$$X(F \circ h) = X^0 \frac{\partial F}{\partial t} + X^i \frac{\partial F}{\partial q^i} + U_i \frac{\partial F}{\partial p_i} - X(H) \frac{\partial F}{\partial p_0},$$

Y will necessarily be of the form

$$Y = X^0 \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + U_i \frac{\partial}{\partial p_i} - X(H) \frac{\partial}{\partial p_0}. \tag{3.8}$$

where, on the right-hand side, we omit the pullback under ρ for functions which come from $J^1\tau^*$.

Lemma 3.4. *$X \in \mathcal{X}(J^1\tau^*)$ and $Y \in \mathcal{X}(T^*E)$ are h-related if and only if Y projects onto X and $Y(\widetilde{H}) = 0$.*

Proof. This is immediately clear from the above coordinate expressions. More intrinsically, Y must project onto X because ρ and h are each others inverse when restricted to the image of h and $Y(\widetilde{H}) = 0$ reflects the fact that Y must be tangent to this image. □

Obviously, $X_{\widetilde{H}}(\widetilde{H}) = 0$, hence X_h and $X_{\widetilde{H}}$ are h-related. But it is certainly not true that the tensor fields $\widetilde{R}$ and $\widetilde{R}_{T^*}$ are also h-related, i.e. that they map general h-related vector fields into h-related vector fields. What is true, however, is that the sequence of vector fields defining the distributions $\mathcal{D}_h$ and $\mathcal{D}_{\widetilde{H}}$ are pairwise h-related.

Lemma 3.5. *The vector fields $\widetilde{R}^k(X_h) \in \mathcal{X}(J^1\tau^*)$ and $\widetilde{R}^k_{T^*}(X_{\widetilde{H}}) \in \mathcal{X}(T^*E)$ are h-related for all k.*

Proof. We already know that the vector fields under consideration are ρ-related. In addition, in view of the fact that $\mathcal{D}_{\widetilde{H}} = \mathcal{D}^{\perp}_{\widetilde{H}}$, we have: $\langle \widetilde{R}^k_{T^*}(X_{\widetilde{H}}), d\widetilde{H}\rangle = 0$ for all k. □

We can now argue that $\mathcal{D}_h$ is a distribution of dimension $n+1$ on $J^1\tau^*$ if we assume that $\widetilde{H}$ and R are such that $\mathcal{D}_{\widetilde{H}}$ has dimension $n+1$ on (an open domain in) T^*E. The linear combination

$$\sum_{k=0}^{n} a_k \widetilde{R}^k(X_h), \qquad a_k \in C^\infty(J^1\tau^*)$$

is h-related to

$$\sum_{k=0}^{n} a_k \widetilde{R}^k_{T^*}(X_{\widetilde{H}}),$$

so the linear independence of the $\widetilde{R}^k_{T^*}(X_{\widetilde{H}})$ implies (pointwise) that the defining vector fields of $\mathcal{D}_h$ are also linearly independent.

Theorem 3.6. *$\mathcal{D}_h$ is an integrable distribution on $J^1\tau^*$ if and only if $\mathcal{D}_{\widetilde{H}}$ is integrable on T^*E.*

Proof. If two vector fields on $J^1\tau^*$ are h-related to corresponding vector fields on T^*E, then so are their Lie brackets. By way of example, consider the pair $(X_h, \widetilde{R}(X_h))$ on $J^1\tau^*$ and the corresponding pair $(X_{\widetilde{H}}, \widetilde{R}_{T^*}(X_{\widetilde{H}}))$ on T^*E, but the reasoning below applies just as well to any other pair. The fact that their brackets are also h-related means that, in terms of the simplified notations used in (3.8), we have

$$\big[X_{\widetilde{H}}, \widetilde{R}_{T^*}(X_{\widetilde{H}})\big] = \big[X_h, \widetilde{R}(X_h)\big] - \big[X_h, \widetilde{R}(X_h)\big](H)\frac{\partial}{\partial p_0}.$$

Now, if $\mathcal{D}_h$ is integrable, we have that $\big[X_h, \widetilde{R}(X_h)\big] = \sum_{k=0}^{n} a_k \widetilde{R}^k(X_h)$ for some functions a_k on $J^1\tau^*$. Using this in the above equality, the right-hand side clearly becomes, again by the formal general rule (3.8), the expression for the h-related vector field $\sum_{k=0}^{n} a_k \widetilde{R}^k_{T^*}(X_{\widetilde{H}})$. This shows that the bracket $\big[X_{\widetilde{H}}, \widetilde{R}_{T^*}(X_{\widetilde{H}})\big]$ belongs to $\mathcal{D}_{\widetilde{H}}$, and similarly for all other pairs, so that $\mathcal{D}_{\widetilde{H}}$ is integrable. Conversely, assume that $\mathcal{D}_{\widetilde{H}}$ is integrable, then $\big[X_{\widetilde{H}}, \widetilde{R}_{T^*}(X_{\widetilde{H}})\big] = \sum_{k=0}^{n} a_k \widetilde{R}^k_{T^*}(X_{\widetilde{H}})$, for some a_k which, in principle, are now functions on T^*E. But all vector fields $\widetilde{R}^k_{T^*}(X_{\widetilde{H}})$ in that sum are h-related to a corresponding element of $\mathcal{D}_h$, so that the previous equality can be rewritten as

$$\big[X_{\widetilde{H}}, \widetilde{R}_{T^*}(X_{\widetilde{H}})\big] = \sum_{k=0}^{n} a_k \widetilde{R}^k_{T^*}(X_{\widetilde{H}}) = \sum_{k=0}^{n} a_k \widetilde{R}^k(X_h) - \sum_{k=0}^{n} a_k \widetilde{R}^k(X_h)(H)\frac{\partial}{\partial p_0}.$$

Identifying now the right-hand sides of the two previous displayed equalities, we conclude that necessarily

$$\big[X_h, \widetilde{R}(X_h)\big] = \sum_{k=0}^{n} a_k \widetilde{R}^k(X_h).$$

The left-hand side in this relation manifestly is a vector field on $J^1\tau^*$, so that there cannot be any p_0-dependence in the overall expression on the right. Therefore, if some of the a_k would explicitly depend on p_0, the partial sum of such terms on the right would have to vanish. But the $\widetilde{R}^k(X_h)$ being linearly independent as vector fields on $J^1\tau^*$, they are also linearly independent as vector fields along the projection $\rho : T^*E \to J^1\tau^*$. This implies that all a_k in that partial sum eventually must vanish. So in the end we will have an equality of the form $\big[X_h, \widetilde{R}(X_h)\big] = \sum_{k=0}^{n} a_k \widetilde{R}^k(X_h)$ with coefficients a_k which, without loss of generality, can all be seen as functions on $J^1\tau^*$. Repeating this argument for all possible brackets of vector fields of the form $\widetilde{R}^k_{T^*}(X_{\widetilde{H}})$ will lead us to the conclusion that also $\mathcal{D}_h$ is integrable. □

In analogy with the case of $\mathcal{D}_F$, one can establish a connection between the integrability of $\mathcal{D}_h$, which is, due to the previous theorem, directly related to the integrability of $\mathcal{D}_{\widetilde{H}}$, and the Hamilton-Jacobi equation for the time-dependent Hamiltonian $H(t, q, p)$. At the end of Section 3.4 we will illustrate this.

By Theorem 3.3, integrability of $\mathcal{D}_{\widetilde{H}}$ is reduced to the condition $dd_{\widetilde{R}_{T^*}}\widetilde{H}\big|_{\mathcal{D}_{\widetilde{H}}} = 0$, and we now know that this will equally ensure integrability of $\mathcal{D}_h$. But there is no doubt that it would be more satisfactory to characterize integrability of $\mathcal{D}_h$ by a condition expressed in terms of objects living on $J^1\tau^*$. This is our final goal for this section and it will be achieved with the aid of a 2-form ω_R defined on $J^1\tau^*$.

Now given a (1,1) tensor field R on E with $R(dt) = 0$, consider the 2-form $\omega_R := dR^h$, with R^h the horizontal lift of R to $J^1\tau^*$ (see Definition 2.20). In coordinates,

$$\begin{aligned}\omega_R &= R^j_i dp_j \wedge dq^i + R^j_0 dp_j \wedge dt \\ &\quad + \frac{1}{2} p_l \left(\frac{\partial R^l_j}{\partial q^k} - \frac{\partial R^l_k}{\partial q^j}\right) dq^k \wedge dq^j + p_l \left(\frac{\partial R^l_0}{\partial q^k} - \frac{\partial R^l_k}{\partial t}\right) dq^k \wedge dt. \qquad (3.9)\end{aligned}$$

Clearly, ω_R is closed. In addition, the assumption about distinct eigenvalues of R implies that $\det(R^i_j) \neq 0$. It is then clear from the above coordinate expression that ω_R has maximal rank everywhere, so that we have a presymplectic structure indeed.

Lemma 3.7. *The presymplectic form ω_R, defined by $\omega_R = dR^h$ has the additional property that*

$$\omega_R = h^*\tau_R^* d\theta_E, \tag{3.10}$$

where τ_R is the fibre linear map on T^*E defined by R,

$$\tau_R : T^*E \to T^*E,\ (t, q^i, p_0, p_i) \mapsto (t, q^i, R^i_0 p_i, R^i_j p_i). \tag{3.11}$$

Proof. It follows easily from the coordinate expression of R^h (2.25) that

$$h^*\tau_R^*\theta_E = h^*\tau_R^*(p_0 dt + p_i dq^i) = R^h.$$

□

Since $\tau_R^* d\theta_E$ is the 2-form needed to define $\widetilde{R}_{T^*}$ (see Definition 2.11), the idea now is to transfer certain properties from T^*E to $J^1\tau^*$ by pulling back via h. Of course, such a pullback works well for forms, but is in general not well defined when it concerns the contraction of a form with an arbitrary vector field. But it does work when the vector fields involved have an h-related companion on $J^1\tau^*$, as we briefly recall first in a general setting.

Let h be a smooth map from a manifold M into a manifold N and let $Y \in \mathcal{X}(N)$ be h-related to $X \in \mathcal{X}(M)$, so that $Th \circ X = Y \circ h$. Then, for any form $\omega \in \bigwedge^k(N)$, we can define $h^*(i_Y\omega) \in \bigwedge^{k-1}(M)$ as follows. For any $m \in M$ and $v_1, \ldots, v_{k-1} \in T_mM$, put

$$\begin{aligned} h^*(i_Y\omega)(m)(v_1, \ldots, v_{k-1}) &= (i_Y\omega)(h(m))(Th(v_1), \ldots, Th(v_{k-1})) \\ &= \omega(h(m))(Th(X_m), Th(v_1), \ldots, Th(v_{k-1})) \\ &= (h^*\omega)(m)(X_m, v_1, \ldots, v_{k-1}), \end{aligned}$$

from which it follows that $h^*(i_Y\omega) = i_X(h^*\omega)$.

Theorem 3.8. *The distribution $\mathcal{D}_h$ on $J^1\tau^*$ is integrable if and only if*

$$\mathcal{L}_{X_h}\omega_R|_{\mathcal{D}_h} = 0. \tag{3.12}$$

Proof. In the course of the proof of Lemma 3.1, applied to the case of $\mathcal{D}_{\widetilde{H}}$, we have seen that

$$i_{\widetilde{R}^k_{T^*}(X_{\widetilde{H}})} d\theta_E = -\widetilde{R}^k_{T^*}(d\widetilde{H}), \qquad \text{for all } k. \tag{3.13}$$

On the other hand, we know from the defining relation (2.6) of $\widetilde{R}_{T^*}$ that

$$i_{\widetilde{R}^k_{T^*}(X_{\widetilde{H}})} d\theta_E = i_{\widetilde{R}^{k-1}_{T^*}(X_{\widetilde{H}})} \tau_R^* d\theta_E. \tag{3.14}$$

It follows that

$$i_{\widetilde{R}^{k-1}_{T^*}(X_{\widetilde{H}})} \tau_R^* d\theta_E = -\widetilde{R}^k_{T^*}(d\widetilde{H}), \qquad k = 1, \ldots, n. \tag{3.15}$$

By Lemma 3.4 about h-related vector fields, we can pull back the relations (3.15) under h, to obtain

$$i_{\widetilde{R}^{k-1}(X_h)} \omega_R = -h^*\Big(\widetilde{R}^k_{T^*}(d\widetilde{H})\Big), \qquad k = 1, \ldots, n.$$

Taking the exterior derivative of this relation for the case $k = 1$ and knowing that ω_R is closed, the result now immediately follows from Theorem 3.3, applied to the case where $F = \widetilde{H}$. This final step of course again relies on the fact that we have bases of $\mathcal{D}_{\widetilde{H}}$ and $\mathcal{D}_h$ consisting of h-related vector fields. □

3.4 Darboux-Nijenhuis coordinates and Forbat's conditions

As announced in the introduction, the integrability of $\mathcal{D}_h$ on $J^1\tau^*$ (or equivalently of $\mathcal{D}_{\widetilde{H}}$ on T^*E) is claimed to be an intrinsic formulation of Forbat's conditions for separability of the time-dependent Hamilton-Jacobi equation. So, we should be able to show that there exists a selection of natural coordinates, such that the condition (3.12) on $J^1\tau^*$ (or equivalently the condition $dd_{\widetilde{R}_{T^*}}\widetilde{H}\big|_{\mathcal{D}_{\widetilde{H}}} = 0$ on T^*E coming from Theorem 3.3) precisely reproduces the conditions (3.1)-(3.2). Needless to say, if such a preferred coordinate system exists, it should have made its appearance in the course of the theoretical developments. It should therefore not come as a surprise that we actually claim that Darboux-Nijenhuis coordinates on $J^1\tau^*$ or T^*E do the job. We shall show this for the condition (3.12), but it can equally well be carried out for the equivalent condition on T^*E.

Suppose we have found the coordinate transformation $(t, q) \to (t, Q(t, q))$ which diagonalizes the tensor field R on E, and let $(t, q, p) \to (t, Q(t, q), P(t, q, p))$ be the induced time-dependent canonical transformation on $J^1\tau^*$. In other words, we have that

$$P_i(t, q, p) = p_l \frac{\partial q^l}{\partial Q^i}(t, Q(t, q)).$$

Then, we know by Theorem 2.37 that $\widetilde{R}$ will take the form

$$\widetilde{R} = \sum_{i=1}^{n} \lambda_i(Q^i) \left(\frac{\partial}{\partial Q^i} \otimes dQ^i + \frac{\partial}{\partial P_i} \otimes dP_i \right).$$

At the same time, the Hamiltonian vector field X_h will have changed its appearance: explicitly, if $H(t, q, p)$ was the Hamiltonian function in the original coordinates, the Hamiltonian $K(t, Q, P)$ in the new coordinates will be given by (from the induced transformation of p_0 on T^*E)

$$K(t, Q, P) = \left[H - p_l \frac{\partial q^l}{\partial t} \right]_{(t,Q,P)}.$$

If we now compute the vector fields $\widetilde{R}^k(X_h)$ spanning the distribution $\mathcal{D}_h$, we readily observe that

$$X_h = \frac{\partial}{\partial t} + \frac{\partial K}{\partial P_i} \frac{\partial}{\partial Q^i} - \frac{\partial K}{\partial Q^i} \frac{\partial}{\partial P_i}$$

and

$$\widetilde{R}^l(X_h) = \lambda_i^l \left(\frac{\partial K}{\partial P_i} \frac{\partial}{\partial Q^i} - \frac{\partial K}{\partial Q^i} \frac{\partial}{\partial P_i} \right), \quad l = 1, \ldots, n.$$

So

$$\begin{pmatrix} X_h \\ \widetilde{R}(X_h) \\ \widetilde{R}^2(X_h) \\ \vdots \\ \widetilde{R}^n(X_h) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ 0 & \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_1^n & \lambda_2^n & \ldots & \lambda_n^n \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial K}{\partial P_1} \frac{\partial}{\partial Q^1} - \frac{\partial K}{\partial Q^1} \frac{\partial}{\partial P_1} \\ \frac{\partial K}{\partial P_2} \frac{\partial}{\partial Q^2} - \frac{\partial K}{\partial Q^2} \frac{\partial}{\partial P_2} \\ \vdots \\ \frac{\partial K}{\partial P_n} \frac{\partial}{\partial Q^n} - \frac{\partial K}{\partial Q^n} \frac{\partial}{\partial P_n} \end{pmatrix}.$$

This strongly suggests a change of basis from $(X_h, \ldots, \widetilde{R}^n(X_h))$ to

$$\left(\frac{\partial}{\partial t}, \ldots, \frac{\partial K}{\partial P_n}\frac{\partial}{\partial Q^n} - \frac{\partial K}{\partial Q^n}\frac{\partial}{\partial P_n}\right),$$

which should no doubt simplify the calculations for the condition (3.12). This is possible since the transition matrix is nonsingular, i.e. its determinant is a multiple of the n-th order Vandermonde determinant and equals

$$(\lambda_1 \ldots \lambda_n) \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix} = (\lambda_1 \ldots \lambda_n) \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$$

which differs from zero because all λ_i are distinct and different form zero as $\lambda_0 = 0$. So we put (no summation over i!)

$$X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial K}{\partial P_i}\frac{\partial}{\partial Q^i} - \frac{\partial K}{\partial Q^i}\frac{\partial}{\partial P_i}, \quad i = 1, \ldots, n \tag{3.16}$$

and observe that this is in fact a set of eigenvectors for the tensor field $\widetilde{R}$, as given by (2.51), in the coordinates under consideration. There is more to say about this observation. Since $\widetilde{R}_{T^*}$ is formally identical to $\widetilde{R}$ in Darboux-Nijenhuis coordinates (see Theorem 2.37), a similar computation of the vector fields $\widetilde{R}^k_{T^*}(X_{\widetilde{H}})$ which span the distribution $\mathcal{D}_{\widetilde{H}}$ on T^*E, will generate via the same nonsingular transition matrix a basis of eigenvectors for $\mathcal{D}_{\widetilde{H}}$, given by (no sum)

$$Y_0 = \frac{\partial}{\partial t} - \frac{\partial K}{\partial t}\frac{\partial}{\partial P_0}, \quad Y_i = \frac{\partial K}{\partial P_i}\frac{\partial}{\partial Q^i} - \frac{\partial K}{\partial Q^i}\frac{\partial}{\partial P_i}, \quad i = 1, \ldots, n. \tag{3.17}$$

It is further interesting to note that the vector fields X_k on $J^1\tau^*$ and Y_k on T^*E are h-related, Y_0 and the Y_i clearly project onto X_0 and X_i respectively and $Y_0(\widetilde{H}) = Y_i(\widetilde{H}) = 0$. Hence, we have proved, by passing to a special selection of coordinates, the following useful addition to Lemma 3.5.

Lemma 3.9. *There exists a transition to local bases for $\mathcal{D}_h$ and $\mathcal{D}_{\widetilde{H}}$ which consist of eigenvectors of $\widetilde{R}$ and $\widetilde{R}_{T^*}$ respectively, and preserves the property that the generating vector fields are h-related.* □

Let us now finally express the condition (3.12) for integrability of $\mathcal{D}_h$ by making use of the new basis of eigenvectors X_k $(k = 0, \ldots, n)$. From (3.9) and (2.50), we see that in the new coordinates, ω_R takes the simple form

$$\omega_R = \sum_{l=1}^{n} \lambda_l(Q^l) dP_l \wedge dQ^l,$$

from which it follows that

$$\mathcal{L}_{X_h}\omega_R = \sum_{l=1}^{n} \frac{\partial K}{\partial P_l}\frac{d\lambda_l}{dQ^l} dP_l \wedge dQ^l + \sum_{l=1}^{n} \lambda_l \left(\frac{\partial^2 K}{\partial Q^l\,\partial t} dQ_l \wedge dt + \frac{\partial^2 K}{\partial P_l\,\partial t} dP_l \wedge dt \right)$$
$$+ \sum_{k,l} (\lambda_l - \lambda_k) \left(\frac{\partial^2 K}{\partial P_l\,\partial Q^k} dP_l \wedge dQ^k + \frac{1}{2}\Big(\frac{\partial^2 K}{\partial Q^k\,\partial Q^l} dQ^l \wedge dQ^k + \frac{\partial^2 K}{\partial P_k\,\partial P_l} dP_l \wedge dP_k \Big) \right).$$

The first term does not contribute anything when acting on the basis of eigenvectors (3.16). From the second term, it follows that

$$\mathcal{L}_{X_h}\omega_R(X_i, X_0) = \lambda_i \left(\frac{\partial^2 K}{\partial Q^i\,\partial t}\frac{\partial K}{\partial P_i} - \frac{\partial^2 K}{\partial P_i\,\partial t}\frac{\partial K}{\partial Q^i} \right) \qquad \text{for fixed } i. \tag{3.18}$$

The last term implies that

$$\begin{aligned}\mathcal{L}_{X_h}\omega_R(X_i, X_j) = (\lambda_j - \lambda_i) \Bigg(&\frac{\partial^2 K}{\partial P_i\,\partial Q^j}\frac{\partial K}{\partial Q^i}\frac{\partial K}{\partial P_j} + \frac{\partial^2 K}{\partial P_j\,\partial Q^i}\frac{\partial K}{\partial Q^j}\frac{\partial K}{\partial P_i} \\ &- \frac{\partial^2 K}{\partial Q^i\,\partial Q^j}\frac{\partial K}{\partial P_i}\frac{\partial K}{\partial P_j} - \frac{\partial^2 K}{\partial P_i\,\partial P_j}\frac{\partial K}{\partial Q^i}\frac{\partial K}{\partial Q^j} \Bigg) \qquad \text{for fixed } i \neq j.\end{aligned} \tag{3.19}$$

Since the λ_i are nonzero and distinct, it is clear now that $\mathcal{L}_{X_h}\omega_R|_{\mathcal{D}_h} = 0$ precisely gives rise to the Forbat conditions (3.1)-(3.2).

We summarize our main results in the following theorem.

Theorem 3.10. *Let E be a bundle over $\mathbb{R}$ of dimension $n+1$. Let h be a section of the bundle $\rho : T^*E \to J^1\tau^*$ and denote by X_h the corresponding Hamiltonian vector field on $J^1\tau^*$. Let R be a type $(1,1)$ tensor field on E with the following properties: (i) $R(dt) = 0$, (ii) $N_R = 0$, (iii) R is algebraically diagonalizable with distinct eigenvalues. Assume further that the $n+1$ vector fields $X_h, \widetilde{R}(X_h), \ldots \widetilde{R}^n(X_h)$ are linearly independent, where $\widetilde{R}$ is the complete lift of R to $J^1\tau^*$. Consider the 2-form $\omega_R = dR^h$, where R^h is the horizontal lift of R to $J^1\tau^*$. Then the distribution*

$\mathcal{D}_h = \operatorname{sp}\{X_h, \widetilde{R}(X_h), \dots \widetilde{R}^n(X_h)\}$ *is integrable provided that* $\mathcal{L}_{X_h}\omega_R|_{\mathcal{D}_h} = 0$. *In Darboux-Nijenhuis coordinates for the Poisson-Nijenhuis structure which* $\widetilde{R}$ *defines on* $J^1\tau^*$, *these integrability conditions are exactly Forbat's necessary and sufficient conditions for separability of a time-dependent Hamilton-Jacobi equation.*

To end this section, we show in a different way, more as an illustration of our theory, that integrability of $\mathcal{D}_h$ leads to a separable solution of the time-dependent Hamilton-Jacobi equation. Now, if $\mathcal{D}_h$ is integrable, we have a foliation of $J^1\tau^*$ with leaves of dimension $n+1$. Every leaf L' of this foliation defines a submanifold $L = h(L')$ of T^*E. The integrability of $\mathcal{D}_h$ implies the integrability of $\mathcal{D}_{\widetilde{H}}$ and since the defining vector fields of both distributions are h-related, the submanifold L will be a leaf of the foliation defined by $\mathcal{D}_{\widetilde{H}}$. Now in the neighbourhood U of a regular point in T^*E, coordinates adapted to the foliation can be introduced: (Q^I, α_I) (with $Q^I = (t, Q^1, \dots, Q^n)$). Following the same reasoning as in Section 3.2 for the Lagrangian distribution $\mathcal{D}_F$ on T^*E, we know that there exists locally a function

$$S_\alpha : \bar{U}_\alpha \subseteq E \to \mathbb{R} \quad \text{such that} \qquad L_\alpha \cap U = dS_\alpha(\bar{U}_\alpha).$$

Thus we can define

$$S : U \subseteq T^*E \to \mathbb{R} : (Q^I, \alpha_I) \mapsto S(Q^I, \alpha_I) = S_\alpha(Q^I).$$

Since $X_{\widetilde{H}}$ is an element of $\mathcal{D}_{\widetilde{H}}$, $\widetilde{H}$ is constant on the leaves of the foliation. So in the adapted coordinates, we can choose α_0 such that $\alpha_0 = \widetilde{H}$. Restricted to the image of h we then have $\alpha_0 = \widetilde{H} = 0$. So we can define

$$\widetilde{S} : \rho(U \cap \operatorname{Im}(h)) \subset J^1\tau^* \to \mathbb{R} : (t, Q^i, \alpha_i) \mapsto \widetilde{S}(t, Q^i, \alpha_i) = S(t, Q^i, 0, \alpha_i).$$

We define $d_1\widetilde{S}$ by

$$d_1\widetilde{S} : \rho(U \cap \operatorname{Im}(h)) \subset J^1\tau^* \to J^1\tau^* : (t, Q^i, \alpha_i) \mapsto d_1\widetilde{S}(t, Q^i, \alpha_i) = \left(t, Q^i, P_i = \frac{\partial \widetilde{S}}{\partial Q^i}\right).$$

Then for $d_1\widetilde{S}$ to be a local diffeomorphism onto its image, the Jacobian should be nonsingular

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\partial Q^i}{\partial Q^j} & \dfrac{\partial Q^i}{\partial \alpha_j} \\ \dfrac{\partial P_i}{\partial t} & \dfrac{\partial P_i}{\partial Q^j} & \dfrac{\partial P_i}{\partial \alpha_j} \end{pmatrix} \neq 0$$

which is equivalent with

$$\det\left(\frac{\partial^2 \widetilde{S}}{\partial Q^i\,\partial \alpha_j}\right) \neq 0.$$

So, the function $\widetilde{S}(t, Q^i, \alpha_i)$ is a complete solution of the time-dependent Hamilton-Jacobi equation associated with $K(t, Q, P)$:

$$\frac{\partial \widetilde{S}}{\partial t} + K\left(t, Q^i, \frac{\partial \widetilde{S}}{\partial Q^i}\right) = 0.$$

It remains to show that $\widetilde{S}$ is separable in Darboux-Nijenhuis coordinates associated to $\widetilde{R}$, i.e. $\widetilde{S}(t, Q^i, \alpha_j) = \widetilde{S}_0(t, \alpha_j) + \widetilde{S}_1(Q^1, \alpha_j) + \ldots + \widetilde{S}_n(Q^n, \alpha_j)$. Now, every leaf of $\mathcal{D}_h$ can locally be characterized by the n equations

$$f_i = P_i - \partial\widetilde{S}/\partial Q^i = 0.$$

The vector fields X spanning the tangent space at each point of the leaf should satisfy $X(f_i) = 0$, $\forall i$. So, the tangent bundle of every leaf of $\mathcal{D}_h$, is locally spanned by the vector fields

$$X_0 = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial^2 \widetilde{S}}{\partial Q^i\,\partial t}\frac{\partial}{\partial P_i}, \qquad X_i = \frac{\partial}{\partial Q^i} + \sum_{j=1}^{n} \frac{\partial^2 \widetilde{S}}{\partial Q^i\,\partial Q^j}\frac{\partial}{\partial P_j}, \qquad i = 1, \ldots, n.$$

The distribution $\mathcal{D}_h$ is by definition $\widetilde{R}$-invariant and therefore the tangent bundle of every leaf of $\mathcal{D}_h$ should be $\widetilde{R}$-invariant. We have

$$\widetilde{R}(X_0) = \sum_{i=1}^{n} \lambda_i \frac{\partial^2 \widetilde{S}}{\partial Q^i\,\partial t}\frac{\partial}{\partial P_i}$$

and this can only be tangent to a leaf of the foliation iff

$$\frac{\partial^2 \widetilde{S}}{\partial Q^i\,\partial t} = 0 \qquad i = 1, \ldots, n. \tag{3.20}$$

On the other hand, for every X_i we have

$$\widetilde{R}(X_i) = \lambda_i \frac{\partial}{\partial Q^i} + \sum_{j=1}^{n} \lambda_j \frac{\partial^2 \widetilde{S}}{\partial Q^i\,\partial Q^j} \frac{\partial}{\partial P_j}$$

and for this to be tangent to a leaf of the foliation it is necessary and sufficient that

$$\frac{\partial^2 \widetilde{S}}{\partial Q^i\,\partial Q^j} = 0 \quad \text{for } j \neq i \tag{3.21}$$

since all eigenfunctions are distinct. From (3.20) and (3.21) it immediately follows that $\widetilde{S}$ is indeed separable.

3.5 Illustrative examples

Having obtained an intrinsic characterization of Hamilton-Jacobi separability in the form of a set of conditions which in principle can be tested in any coordinate system, i.e. prior to knowing separation coordinates, the next challenging question is of course: "What is the practical content of these conditions?". This certainly is an interesting topic for further research. At this moment, the best we can do is to make a certain ansatz about the kind of (1,1) tensor field R which is good candidate for discovering new cases of separable Hamiltonians. The database of separable systems, developed in a systematic way in [12], can be a good starting point in the search for generalizations. We illustrate this in the following 2-dimensional example.

Consider a time-dependent Hamiltonian of the form

$$\begin{aligned} H = {} & \tfrac{1}{2} f_1(t) p_1^2 + \tfrac{1}{2} f_2(t) p_2^2 + c_1(t) p_1 q_1 + c_2(t) p_2 q_2 \\ & + a_1(t) q_1^3 + a_2(t) q_2^3 + a_3(t) q_1^2 q_2 + a_4(t) q_1 q_2^2 + a_5(t) q_1^2 + a_6(t) q_2^2 + a_7(t) q_1 q_2. \end{aligned} \tag{3.22}$$

This Hamiltonian contains eleven as yet arbitrary functions of time, but we assume that $f_1(t) \neq 0$ and $f_2(t) \neq 0$. [For clarity, we use lower indices for the q-variables in such explicit polynomial expressions.] The point about (3.22) is that one can verify that at least there exist particular selections for these arbitrary functions, for which we encounter (after a coordinate transformation) a known separable case described in [12]. Likewise, we make an ansatz about the tensor field R: we take it of the form

$$R = \rho(t) f(q_1, q_2) \left(\frac{\partial}{\partial q_1} \otimes dq_1 + \frac{\partial}{\partial q_2} \otimes dq_2 \right) + g(q_1, q_2) \left(\sigma_1(t) \frac{\partial}{\partial q_1} \otimes dq_2 \right.$$
$$\left. + \sigma_2(t) \frac{\partial}{\partial q_2} \otimes dq_1 \right) + R^1_0(t, q_1, q_2) \frac{\partial}{\partial q_1} \otimes dt + R^2_0(t, q_1, q_2) \frac{\partial}{\partial q_2} \otimes dt. \quad (3.23)$$

The matrix R^i_j contains three arbitrary functions of time and two arbitrary functions depending on q_1 and q_2, the R^i_0-components are arbitrary functions of time and the q-variables.

First of all, the eigenfunctions of the matrix R^i_j are given by

$$\lambda_i = \rho(t) f(q_1, q_2) \pm \sqrt{\sigma_1(t)\sigma_2(t)}\, g(q_1, q_2). \quad (3.24)$$

Since we want R to have distinct eigenvalues, we have to require that none of the $\sigma_i(t)$ or $g(q_1, q_2)$ becomes zero. Of course we consider only the domain where $\sigma_1(t)\sigma_2(t) > 0$.

The first condition we impose now is that R should have vanishing Nijenhuis torsion. In our case, since $R(dt) = 0$, this specifically means that the following partial differential equations should be satisfied (with partial derivatives denoted by a comma, and summation over k):

$$R^i_k (R^k_{\beta,\gamma} - R^k_{\gamma,\beta}) = R^k_\gamma R^i_{\beta,k} - R^k_\beta R^i_{\gamma,k} \qquad i = 1, 2 \text{ and } \beta, \gamma = 0, 1, 2.$$

The conditions which do not yet involve the R^i_0-components are

$$\rho(t) \frac{\partial f}{\partial q_1} - \sigma_2(t) \frac{\partial g}{\partial q_2} = 0 \qquad \text{and} \qquad \rho(t) \frac{\partial f}{\partial q_2} - \sigma_1(t) \frac{\partial g}{\partial q_1} = 0. \quad (3.25)$$

The other conditions read (summation over k):

$$R^i_k (R^k_{0,j} - R^k_{j,t}) = R^k_j R^i_{0,k} - R^k_0 R^i_{j,k} \qquad i, j = 1, 2. \quad (3.26)$$

This is a set of four partial differential equations for the R^i_0.

To start, let us take a look at the conditions (3.25). Because the σ_i are different from zero, we can rewrite them as

$$\frac{\partial g}{\partial q_2} = \frac{\rho(t)}{\sigma_2(t)} \frac{\partial f}{\partial q_1} \qquad \text{and} \qquad \frac{\partial g}{\partial q_1} = \frac{\rho(t)}{\sigma_1(t)} \frac{\partial f}{\partial q_2}. \quad (3.27)$$

The left-hand sides of these equations do not depend on t, so also the right-hand sides can not depend on time

$$\frac{d}{dt}\left(\frac{\rho(t)}{\sigma_2(t)}\right)\frac{\partial f}{\partial q_1} = \frac{d}{dt}\left(\frac{\rho(t)}{\sigma_1(t)}\right)\frac{\partial f}{\partial q_2} = 0.$$

There are several possible solutions.

1. The first possibility is that both t-factors are zero, i.e.

$$\frac{\rho(t)}{\sigma_2(t)} = k_1 \qquad \text{and} \qquad \frac{\rho(t)}{\sigma_1(t)} = k_2, \qquad k_1, k_2 \in \mathbb{R}.$$

The integrability conditions of (3.27) then imply

$$k_1 \frac{\partial^2 f}{\partial q_1^2} = k_2 \frac{\partial^2 f}{\partial q_2^2}. \tag{3.28}$$

We make a distinction between $k_1 = 0$ and $k_1 \neq 0$.

a. Suppose $k_1 \neq 0$. If we assume that $k_2/k_1 = c^2$, we get the wave equation for (3.28)

$$\frac{\partial^2 f}{\partial q_1^2} - c^2 \frac{\partial^2 f}{\partial q_2^2} = 0.$$

Hence $f = f_1(u) + f_2(v)$ with $u = q_2 + cq_1$, $v = q_2 - cq_1$ and f_1, f_2 arbitrary functions of u, v respectively. We also must have

$$k_1 \frac{\partial^2 g}{\partial q_1^2} = k_2 \frac{\partial^2 g}{\partial q_2^2}$$

hence $g = g_1(u) + g_2(v)$ with, based on (3.27),

$$g_1'(u) + g_2'(v) = k_1 c(f_1'(u) - f_2'(v))$$
$$c(g_1'(u) - g_2'(v)) = k_2(f_1'(u) + f_2'(v)).$$

Frow which it follows that $f_1'(u) = c/k_2 g_1'$ and $f_2'(v) = -c/k_2 g_2'$ ($k_2 \neq 0$, otherwise this would imply $\rho(t) = 0$ and so $k_1 = 0$ which is ruled out), or thus

$$f = c/k_2(g_1 - g_2) + l_1, \qquad l_1 \in \mathbb{R}.$$

   b. If $k_1 = 0$, it immediately follows that $\rho(t) = 0$. Then (3.25) implies that g must be constant, $g = c$.

2. Secondly, both q-factors can be zero, so $f = a$, $a \in \mathbb{R}$. Then (3.25) implies that also g is constant, $g = b \in \mathbb{R}$.

3. At last, we can have a combination of the previous assumptions. Take for example $\partial f/\partial q_2 = 0$ and $\rho(t)/\sigma_2(t) = k_1$. [Taking $\partial f/\partial q_1 = 0$ and $\rho(t)/\sigma_1(t) = k_2$ would generate, after renaming the q-variables, the same solution.] Then $f = aq_1 + b_1$ and from (3.25) it follows that $g = k_1 a q_2 + b_2$, with $a, b_1, b_2 \in \mathbb{R}$.

Now, for internal logic, we should keep, in case 1, f and g function of q_1 and q_2, in case 2, ρ/σ_2 and ρ/σ_1 functions of time and in case 3, $f = f(q_1)$ (or $a \neq 0$) and ρ/σ_1 not constant.

For these four different cases we can now solve the remaining conditions (3.26) for $N_R = 0$.

1a. In this case, the four partial differential equations (3.26) for the R_0^i are

$$k_1(g_1+g_2)\left(-c^2\frac{\partial R_0^1}{\partial q_2}+\frac{\partial R_0^2}{\partial q_1}\right)+c^2k_1(g_1'+g_2')R_0^1+ck_1(g_1'-g_2')R_0^2 \\ +\dot{\rho}(-2g_1^2-2g_2^2-2ck_1l_1(g_1-g_2)-c^2k_1^2l_1^2)=0, \tag{3.29}$$

$$ck_1(g_1+g_2)\left(\frac{\partial R_0^1}{\partial q_1}-\frac{\partial R_0^2}{\partial q_2}\right)+c^2k_1(g_1'-g_2')R_0^1+ck_1(g_1'+g_2')R_0^2 \\ +\dot{\rho}(-2g_1^2+2g_2^2-2ck_1l_1(g_1+g_2))=0, \tag{3.30}$$

$$-ck_1(g_1+g_2)\left(\frac{\partial R_0^1}{\partial q_1}-\frac{\partial R_0^2}{\partial q_2}\right)+c^2k_1(g_1'-g_2')R_0^1+ck_1(g_1'+g_2')R_0^2 \\ +\dot{\rho}(-2g_1^2+2g_2^2-2ck_1l_1(g_1+g_2))=0, \tag{3.31}$$

$$k_1(g_1+g_2)\left(c^2\frac{\partial R_0^1}{\partial q_2}-\frac{\partial R_0^2}{\partial q_1}\right)+c^2k_1(g_1'+g_2')R_0^1+ck_1(g_1'-g_2')R_0^2 \\ +\dot{\rho}(-2g_1^2-2g_2^2-2ck_1l_1(g_1-g_2)-c^2k_1^2l_1^2)=0. \tag{3.32}$$

Taking the difference of (3.29) and (3.32), and of (3.30) and (3.31), we get

$$k_1(g_1+g_2)\left(-c^2\frac{\partial R_0^1}{\partial q_2}+\frac{\partial R_0^2}{\partial q_1}\right)=0,$$
$$ck_1(g_1+g_2)\left(\frac{\partial R_0^1}{\partial q_1}-\frac{\partial R_0^2}{\partial q_2}\right)=0.$$

Since $g=g_1+g_2\neq 0$, it follows that

$$R_0^1=\frac{\partial F(t,q)}{\partial q_2},\qquad R_0^2=\frac{\partial F(t,q)}{\partial q_1}\qquad\text{and}\qquad \frac{\partial^2 F}{\partial q_1^2}-c^2\frac{\partial^2 F}{\partial q_2^2}=0.$$

So

$$R_0^1=\frac{\partial F_1(t,u)}{\partial u}+\frac{\partial F_2(t,v)}{\partial v},\qquad R_0^2=c\left(\frac{\partial F_1(t,u)}{\partial u}-\frac{\partial F_2(t,v)}{\partial v}\right),$$

where we defined u and v already: $u=q_2+cq_1$, $v=q_2-cq_1$. Then (3.29) and (3.30) imply

$$\begin{aligned}\dot\rho(-2g_1^2-2g_2^2-2ck_1l_1(g_1-g_2)-c^2k_1^2l_1^2)&\\ +2c^2k_1\left(g_1'\frac{\partial F_1(t,u)}{\partial u}+g_2'\frac{\partial F_2(t,v)}{\partial v}\right)&=0,\\ \dot\rho(-2g_1^2+2g_2^2-2ck_1l_1(g_1+g_2))&\\ +2c^2k_1\left(g_1'\frac{\partial F_1(t,u)}{\partial u}-g_2'\frac{\partial F_2(t,v)}{\partial v}\right)&=0.\end{aligned}$$

Taking the sum and the difference of the previous equations, we get

$$4c^2k_1h(t)g_1'\frac{\partial F_1(t,u)}{\partial u}+\dot\rho(-4g_1^2-4ck_1l_1g_1-c^2k_1^2l_1^2)=0,$$
$$4c^2k_1h(t)g_2'\frac{\partial F_2(t,v)}{\partial v}+\dot\rho(-4g_2^2+4ck_1l_1g_2-c^2k_1^2l_1^2)=0.$$

Then, if $g_1'\neq 0$ and $g_2'\neq 0$,

$$R_0^1=\dot\rho\left(\frac{4g_1^2+4ck_1l_1g_1+c^2k_1^2l_1^2}{4c^2k_1g_1'}+\frac{4g_2^2-4ck_1l_1g_2+c^2k_1^2l_1^2}{4c^2k_1g_2'}\right),\tag{3.33}$$
$$R_0^2=\dot\rho\left(\frac{4g_1^2+4ck_1l_1g_1+c^2k_1^2l_1^2}{4ck_1g_1'}-\frac{4g_2^2-4ck_1l_1g_2+c^2k_1^2l_1^2}{4ck_1g_2'}\right).\tag{3.34}$$

If $g_1' = 0$ (or similarly $g_2' = 0$), then

$$R_0^1 = \frac{\partial F_1(t,u)}{\partial u} + \dot\rho\,\frac{4g_2^2 - 4ck_1l_1g_2 + c^2k_1^2l_1^2}{4c^2k_1g_2'},$$
$$R_0^2 = c\frac{\partial F_1(t,u)}{\partial u} - \dot\rho\,\frac{4g_2^2 - 4ck_1l_1g_2 + c^2k_1^2l_1^2}{4ck_1g_2'}.$$

Note that g_1' and g_2' can not both be zero because this would imply that g is constant and thus f constant or $\rho = 0$, which are both part of another (sub)case.

1b. Now, the partial differential equations (3.26) for the R_0^i reduce to

$$c\sigma_1\dot\sigma_2 - \sigma_1\frac{\partial R_0^2}{\partial q_1} + \sigma_2\frac{\partial R_0^1}{\partial q_2} = 0, \tag{3.35}$$
$$\sigma_2\left(\frac{\partial R_0^2}{\partial q_2} - \frac{\partial R_0^1}{\partial q_1}\right) = 0, \tag{3.36}$$
$$\sigma_1\left(\frac{\partial R_0^2}{\partial q_2} - \frac{\partial R_0^1}{\partial q_1}\right) = 0, \tag{3.37}$$
$$c\sigma_2\dot\sigma_1 + \sigma_1\frac{\partial R_0^2}{\partial q_1} - \sigma_2\frac{\partial R_0^1}{\partial q_2} = 0. \tag{3.38}$$

It immediately follows that $\sigma_1\sigma_2$ must be constant, say k^2, and $R_0^1 = \partial F(t,q)/\partial q_2$ and $R_0^2 = \partial F(t,q)/\partial q_1$. Then we must have

$$\frac{\partial^2 F}{\partial q_1^2} - \frac{\sigma_2}{\sigma_1}\frac{\partial^2 F}{\partial q_2^2} = c\dot\sigma_2. \tag{3.39}$$

This is a non-homogenous partial differential equation for F, with particular solution $F = \frac{1}{2}c\dot\sigma_2 q_1^2$. Then, to avoid square roots, assume that $\sigma_2/\sigma_1 = \sigma(t)^2$, so that the general solution is

$$F(t,q) = h_1(t,w) + h_2(t,z) + \tfrac{1}{2}c\dot\sigma_2 q_1^2,$$

with $w = q_2 + \sigma(t)q_1$ and $z = q_2 - \sigma(t)q_1$. Hence

$$R_0^1 = \frac{\partial h_1(t,w)}{\partial w} + \frac{\partial h_2(t,z)}{\partial z}, \tag{3.40}$$
$$R_0^2 = \sigma(t)\left(\frac{\partial h_1(t,w)}{\partial w} - \frac{\partial h_2(t,z)}{\partial z}\right) + kc\dot\sigma q_1 \tag{3.41}$$

as it follows from $\sigma_1\sigma_2 = k^2$ and $\sigma_2/\sigma_1 = \sigma(t)^2$ that $\sigma_2 = k\sigma$ and $\sigma_1 = k/\sigma$.

2. First note that we can exclude the case $f = a = 0$ since this would generate the same tensor field R as in case 1b where $\rho(t) = 0$. For (3.26) we get

$$-b^2\sigma_1\dot{\sigma}_2 - a^2\rho\dot{\rho} - b\sigma_2\frac{\partial R_0^1}{\partial q_2} + b\sigma_1\frac{\partial R_0^2}{\partial q_1} = 0, \tag{3.42}$$

$$a\sigma_2\dot{\rho} + a\rho\dot{\sigma}_2 + \sigma_2\left(\frac{\partial R_0^2}{\partial q_2} - \frac{\partial R_0^1}{\partial q_1}\right) = 0, \tag{3.43}$$

$$a\sigma_1\dot{\rho} + a\rho\dot{\sigma}_1 + \sigma_1\left(\frac{\partial R_0^1}{\partial q_1} - \frac{\partial R_0^2}{\partial q_2}\right) = 0, \tag{3.44}$$

$$-b^2\sigma_2\dot{\sigma}_1 - a^2\rho\dot{\rho} + b\sigma_2\frac{\partial R_0^1}{\partial q_2} - b\sigma_1\frac{\partial R_0^2}{\partial q_1} = 0. \tag{3.45}$$

Rewrite (3.42) and (3.43) as follows

$$b\sigma_2\frac{\partial R_0^1}{\partial q_2} = b\sigma_1\frac{\partial R_0^2}{\partial q_1} - b^2\sigma_1\dot{\sigma}_2 - a^2\rho\dot{\rho}, \tag{3.46}$$

$$\sigma_2\frac{\partial R_0^1}{\partial q_1} = \sigma_2\frac{\partial R_0^2}{\partial q_2} + a\sigma_2\dot{\rho} + a\rho\dot{\sigma}_2. \tag{3.47}$$

The integrability conditions for R_0^1 then imply

$$\frac{\partial^2 R_0^2}{\partial q_1^2} - \frac{\sigma_2}{\sigma_1}\frac{\partial^2 R_0^2}{\partial q_2^2} = 0,$$

with general solution, if we assume that $\sigma_2/\sigma_1 = \sigma(t)^2$,

$$R_0^2 = h_1(t,w) + h_2(t,z) \qquad \text{where} \quad w = q_2 + \sigma(t)q_1,\ z = q_2 - \sigma(t)q_1. \tag{3.48}$$

Then (3.46) and (3.47) determine R_0^1

$$R_0^1 = \frac{1}{\sigma}(h_1(t,w) - h_2(t,z)) - \frac{b^2\sigma_1\dot{\sigma}_2 + a^2\rho\dot{\rho}}{b\sigma_2}q_2 + \frac{a(\sigma_2\dot{\rho} + \rho\dot{\sigma}_2)}{\sigma_2}q_1 + f(t), \tag{3.49}$$

with $f(t)$ an arbitrary function of time. Now (3.44) and (3.45) imply

$$2\sigma_1\sigma_2\dot{\rho} + \rho(\dot{\sigma}_2\sigma_1 + \dot{\sigma}_1\sigma_2) = 0,$$
$$b^2(\dot{\sigma}_2\sigma_1 + \dot{\sigma}_1\sigma_2) + 2a^2\rho\dot{\rho} = 0.$$

This results in

$$\sigma_1\sigma_2 = k^2, \qquad \rho(t) = c, \qquad k, c \in \mathbb{R}.$$

Hence, since $\sigma_2/\sigma_1 = \sigma(t)^2$, $\sigma_1 = k/\sigma$ and $\sigma_2 = k\sigma$. This way R_0^1 simplifies to

$$R_0^1 = \frac{1}{\sigma}(h_1(t,w) - h_2(t,z)) - \frac{bk\dot{\sigma}}{\sigma^2}q_2 + \frac{ac\dot{\sigma}}{\sigma}q_1 + f(t). \tag{3.50}$$

3. The four partial differential equations for the R_0^i (3.26) reduce to

$$\sigma_1\dot{\rho}(ak_1q_2+b_2)^2 + (ak_1q_2+b_2)\left(\rho\frac{\partial R_0^1}{\partial q_2} - k_1\sigma_1\frac{\partial R_0^2}{\partial q_1}\right) + k_1\rho\dot{\rho}(aq_1+b_1)^2 - ak_1\rho R_0^1 = 0, \tag{3.51}$$

$$2\dot{\rho}(aq_1+b_1)(ak_1q_2+b_2) + (ak_1q_2+b_2)\left(\frac{\partial R_0^2}{\partial q_2} - \frac{\partial R_0^1}{\partial q_1}\right) - ak_1R_0^2 = 0, \tag{3.52}$$

$$(\sigma_1\dot{\rho}+\dot{\sigma}_1\rho)(aq_1+b_1)(ak_1q_2+b_2) + (ak_1q_2+b_2)\sigma_1\left(\frac{\partial R_0^1}{\partial q_1} - \frac{\partial R_0^2}{\partial q_2}\right) - ak_1\sigma_1R_0^2 = 0, \tag{3.53}$$

$$\rho\dot{\sigma}_1(ak_1q_2+b_2)^2 + (ak_1q_2+b_2)\left(k_1\sigma_1\frac{\partial R_0^2}{\partial q_1} - \rho\frac{\partial R_0^1}{\partial q_2}\right) + k_1\rho\dot{\rho}(aq_1+b_1)^2 - ak_1\rho R_0^1 = 0. \tag{3.54}$$

In this case the R_0^i can be determined from algebraic relations. Taking the sum of (3.51) and (3.54) we get

$$2k_1\rho\dot{\rho}(aq_1+b_1)^2 + (\sigma_1\dot{\rho}+\dot{\sigma}_1\rho)(ak_1q_2+b_2)^2 - 2ak_1\rho R_0^1 = 0,$$

from which follows

$$R_0^1 = \frac{\dot{\rho}}{a}(aq_1+b_1)^2 + \frac{\sigma_1\dot{\rho}+\dot{\sigma}_1\rho}{2ak_1\rho}(ak_1q_2+b_2)^2.$$

Similarly, for the sum of (3.52) and (3.53), we get

$$(3\sigma_1\dot{\rho}+\dot{\sigma}_1\rho)(aq_1+b_1)(ak_1q_2+b_2) - 2ak_1\sigma_1R_0^2 = 0,$$

such that

$$R_0^2 = \frac{3\sigma_1\dot{\rho}+\dot{\sigma}_1\rho}{2ak_1\sigma_1}(aq_1+b_1)(ak_1q_2+b_2). \tag{3.55}$$

It remains to impose (3.51) and (3.52) (or equivalently (3.53) and (3.54)), but this is identically satisfied.

We can now compute the vector fields $X_h, \widetilde{R}(X_h), \widetilde{R}^2(X_h)$ spanning the distribution $\mathcal{D}_h$, and the 2-form $\mathcal{L}_{X_h}\omega_R$. Imposing the requirement $\mathcal{L}_{X_h}\omega_R|_{\mathcal{D}_h} = 0$ is a fairly straightforward matter now: it gives rise to polynomial expressions in the (q,p)-variables, the coefficients of which all have to vanish. Nevertheless, these computations are tedious so that assistance of Maple (or any other computer algebra package) is a great asset. But even then, the calculations are hard. So, as it is only a matter of illustrating our theory, we will limit ourselves to finding particular solutions for all different cases in such a way that the computations are doable. We will not give a full account of the calculations involved, but merely indicate the order in which consecutive information is gathered, which will ultimately lead to the identification of admissible Hamiltonians of the form (3.22).

3.5.1 Case 1a

First of all take $l_1 = 0$ and $k_1 = k_2 = 1$ such that $c = 1$. Moreover we make an ansatz concerning the functions g_1 and g_2. Taking both g_1 and g_2 constant would be the simplest, but this is ruled out since it would imply that g is constant, which is part of case 2. So let us take, for example, $g_1 = 2u$ and $g_2 = v$, this way g (and f) is function of both q_1 and q_2 and also the R^i_0 are completely determined. The introduction of u and v suggest the coordinate transformation $(t, q_1, q_2) \to (t, u, v)$, with $u = q_2 + q_1$, $v = q_2 - q_1$, on E. In the new coordinates (t, u, v), R is given by

$$R = 2\rho(t)u\frac{\partial}{\partial u} \otimes du - \rho(t)v\frac{\partial}{\partial v} \otimes dv + 2\dot\rho(t)u^2\frac{\partial}{\partial u} \otimes dt - \dot\rho(t)v^2\frac{\partial}{\partial v} \otimes dt. \tag{3.56}$$

Let $(t, q, p) \to (t, u, v, p_u, p_v)$, $p_u = \frac{1}{2}(p_1 + p_2)$ and $p_v = \frac{1}{2}(p_2 - p_1)$, be the induced canonical transformation on $J^1\tau^*$. The Hamiltonian in the new coordinates is

$$\begin{aligned} K = {} & \tfrac{1}{2}F_1(t)(p_u^2 + p_v^2) + F_2(t)p_u p_v + C_1(t)(up_u + vp_v) + C_2(t)(vp_u + up_v) \\ & + A_1(t)u^3 + A_2(t)v^3 + A_3(t)u^2v + A_4(t)uv^2 + A_5(t)u^2 + A_6(t)v^2 + A_7(t)uv, \end{aligned} \tag{3.57}$$

still containing eleven arbitrary functions of time.

Now, the condition $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ gives rise to a polynomial expression of degree 2 in the p-variables; the coefficients are themselves polynomials in u and v, with a degree which varies from 2 to 6. It turns out that the degree 2 terms in the p-variables generate, among other conditions, algebraic relations of the form

$$A_3(t)\rho(t)^2F_1(t)F_2(t) = 0, \qquad A_4(t)\rho(t)^2F_1(t)F_2(t) = 0.$$

$\rho(t) = 0$ is ruled out and, since we are only looking for a particular solution, assume that $F_1(t) \neq 0$ and $F_2(t) \neq 0$, such that the previous conditions imply that

$$A_3(t) = 0, \qquad A_4(t) = 0.$$

In this case, the degree 2 terms in the p-variables also pin down $A_7(t)$,

$$A_7(t) = \frac{C_2(t)^2}{F_2(t)}.$$

It then follows that the remaining coefficients of p_u^2 and p_v^2, which are ODEs, can be solved for $C_1(t)$ and $C_2(t)$

$$C_1(t) = c_1 F_1(t)\rho(t)^2 - \frac{\dot{\rho}(t)}{\rho(t)}, \qquad C_2(t) = c_2 F_1(t)\rho(t)^2, \qquad c_1, c_2 \in \mathbb{R}.$$

From the coefficient of uvp_up_v it follows that $c_1 = c_2$. Further useful info comes from the coefficient of p_u,

$$\begin{aligned} A_1(t) &= \alpha_1 F_1(t)\rho(t)^5, \qquad & A_2(t) &= \alpha_2 F_2(t)\rho(t)^4, \\ A_5(t) &= \alpha_5 F_1(t)\rho(t)^4, \qquad & A_6(t) &= \alpha_6 F_2(t)\rho(t)^4 + \tfrac{1}{2}c_2 F_1(t)\rho(t)^4, \end{aligned}$$

where α_1, α_2, α_5 and α_6 are arbitrary constants. By fixing for example $F_2(t)$, $F_2(t) = m_1 F_1(t)$ ($m_1 \in \mathbb{R}$), the coefficient of p_v vanishes. Finally, to make sure that also the constant term in the p-variables, which is a polynomial expression in u and v of degree 6, is zero, a possible solution is

$$\alpha_1 = 0, \alpha_5 = \tfrac{1}{2}c_2^2.$$

This guarantees that $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$, but there are two more polynomial relations to be investigated, coming from the requirements $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0$ and $\mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$. It is an intriguing observation, however, that these are identically satisfied as a result of the conclusions we drew from the first condition. We thus arrive at the following class of separable Hamiltonians,

$$\begin{aligned} K = \tfrac{1}{2}F_1(t)(p_u^2 + p_v^2 + m_1 p_u p_v) + C_1(t)(up_u + vp_v) + C_2(t)(vp_u + up_v) \\ + A_2(t)v^3 + A_5(t)u^2 + A_6(t)v^2 + A_7(t)uv \end{aligned} \tag{3.58}$$

and the function ρ is still free so the R-tensor which guarantees separability is given by (3.56). Let us finally determine coordinates in which the Hamiltonian is separable.

The coefficients of R in Darboux-Nijenhuis coordinates can be arbitrary functions $\lambda_i(Q^i)$, each depending on a single coordinate Q^i. Hence, they can be chosen to be the Q^i themselves. The structure of the R^i_j in (3.56) therefore already tells us what the linear change of coordinates is,

$$Q_1 = 2\rho(t)u, \qquad Q_2 = -\rho(t)v$$

with induced transformation

$$P_1 = \frac{p_u}{2\rho(t)}, \qquad P_2 = -\frac{p_v}{\rho(t)}.$$

The Hamiltonian of the transformed system is given by

$$\begin{aligned}\widetilde{K} = \rho(t)^2 F_1(t) \big(&2P_1^2 + \tfrac{1}{2}P_2^2 - 2m_1P_1P_2 + c_2(Q_1P_1 + Q_2P_2) - c_2m_1(2Q_2P_1 + Q_1P_2) \\ &+\tfrac{1}{8}c_2^2Q_1^2 - \tfrac{1}{2}c_2^2m_1Q_1Q_2 - \alpha_2m_1Q_2^3 + \tfrac{1}{2}(c_2^2 + 2\alpha_6m_1)Q_2^2\big)\,. \end{aligned} \tag{3.59}$$

and R is given by

$$R = Q_1\frac{\partial}{\partial Q_1} \otimes dQ_1 + Q_2\frac{\partial}{\partial Q_2} \otimes dQ_2. \tag{3.60}$$

It can easily be checked that the Hamilton-Jacobi equation for $\widetilde{K}$ can be solved by separation of variables indeed.

3.5.2 Case 1b

Let us start by taking $k = 1$ and $c = 1$ and for example $h_1(t, w) = w$ and $h_2(t, z) = z$. The introduction of w and z suggest the time-dependent coordinate transformation $(t, q_1, q_2) \to (t, w, z)$, with $w = q_2 + \sigma(t)q_1$, $z = q_2 - \sigma(t)q_1$, on E. In the new coordinates (t, w, z), R is given by

$$R = \frac{\partial}{\partial w} \otimes dw - \frac{\partial}{\partial z} \otimes dz + 2\sigma(t)\frac{\partial}{\partial w} \otimes dt - 2\sigma(t)\frac{\partial}{\partial z} \otimes dt. \tag{3.61}$$

Consider the induced time-dependent canonical transformation on $J^1\tau^*$,

$$p_w = \frac{1}{2\sigma}p_1 + \frac{1}{2}p_2, \qquad p_z = -\frac{1}{2\sigma}p_1 + \frac{1}{2}p_2.$$

The Hamiltonian K in the new coordinates is then given by

$$K = \tfrac{1}{2}F_1(t)(p_w^2 + p_z^2) + F_2(t)p_w p_z + C_1(t)(wp_w + zp_z) + C_2(t)(zp_w + wp_z) \\ + A_1(t)w^3 + A_2(t)z^3 + A_3(t)w^2 z + A_4(t)wz^2 + A_5(t)w^2 + A_6(t)z^2 + A_7(t)wz. \tag{3.62}$$

The condition $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ gives rise to a polynomial expression of degree 2 in p_w and p_z; the coefficients are themselves polynomials in w and z, with a degree which varies from 0 to 4. It turns out that the degree 2 terms in the p-variables generate, among other conditions, algebraic relations of the form

$$A_3(t)F_1(t)F_2(t) = 0, \qquad A_4(t)F_1(t)F_2(t) = 0,$$
$$A_3(t)(F_1(t)^2 + F_2(t)^2) = 0, \qquad A_4(t)(F_1(t)^2 + F_2(t)^2) = 0.$$

We don't want $F_1(t)$ and $F_2(t)$ both to be zero, so this implies that

$$A_3(t) = 0, \qquad A_4(t) = 0.$$

From the constant term in $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ follows that

$$A_7(t) = A_5(t) + A_6(t).$$

All the remaining coefficients of $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ are differential equations. But from the other conditions, $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0$ and $\mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$, we find some more algebraic relations. The constant term of $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0$ implies that

$$A_5(t) = A_6(t).$$

The coefficients of the constant term in the p-variables, but linear in w or z then fixes A_1 and A_2,

$$A_1(t) = A_2(t) = \frac{A_6(t)(C_2(t) - C_1(t))}{3\sigma(t)}.$$

The coefficient of p_w^2 and p_z^2 in $\mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$ is then given by

$$F_1(t)(F_2(t)A_6(t) - C_2(t)^2),$$

for this to be zero there are 2 possibilities (if we assume $F_1(t) \neq 0$), either $F_2(t) = 0$ and $C_2(t) = 0$ or $A_6(t) = C_2(t)^2/F_2(t)$.

Subcase 1: $F_2(t) = 0$ **and** $C_2(t) = 0$

We further find that also $A_6(t) = 0$. The coefficient of p_w^2 in $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0$ is then a first-order differential equation for C_1, with solution

$$C_1(t) = c_1 F_1(t), \qquad c_1 \in \mathbb{R}.$$

The only remaining equation to be solved (except for the case $c_1 = 0$, but this would generate a subcase of a case discussed below) fixes $F_1(t)$,

$$F_1(t) = \frac{\sigma(t)}{c_1 \int \sigma(t) dt}.$$

We arrive at the following class of separable Hamiltonians,

$$K = \frac{\sigma(t)}{c_1 \int \sigma(t) dt}\left(\tfrac{1}{2}(p_w^2 + p_z^2) + c_1(wp_w + zp_z)\right). \tag{3.63}$$

However, K already satisfies Forbat's conditions in the w, z-coordinates, so there is no need for introducing Darboux-Nijenhuis coordinates associated with R.

Subcase 2: $A_6(t) = C_2(t)^2/F_2(t)$

Combining two algebraic relations (the coefficient of $p_w p_z$ in $\mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$ and the coefficient of w^2z^2 in $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$)

$$F_1(t)C_2(t) - F_2(t)C_1(t) = 0, \qquad C_2(t)(C_2(t) - C_1(t)) = 0,$$

we find two possible solutions,

$$C_1(t) = C_2(t) = 0, \quad \text{or} \quad C_2(t) = C_1(t) \text{ and } F_1(t) = F_2(t).$$

In the second case, the remaining coefficients of $\mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$ and $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ imply that the only possible solution is $C_2(t) = 0$. This way, the second solution is in fact a subcase of the first one. In the first case, we find a class of separable Hamiltonians

$$K = \tfrac{1}{2}F_1(t)(p_w^2 + p_z^2) + F_2(t)p_w p_z \tag{3.64}$$

which are again already separable in the w, z-coordinates.

So, the only solutions we find for K are already separable in the w, z-coordinates. We can ask ourselves the question whether it is possible to find a more general separable potential for the Hamiltonian (3.64), which is not yet separable in the w, z-coordinates. So, let us now take a Hamiltonian of the form

$$K = \tfrac{1}{2}F_1(t)(p_w^2 + p_z^2) + F_2(t)p_w p_z + V(t, w, z).$$

and search for a particular solution for V such that $\mathcal{L}_{X_h}\omega_R|_{\mathcal{D}_h} = 0$. The requirement $\mathcal{L}_{X_h}\omega_R|_{\mathcal{D}_h} = 0$ now gives rise to polynomial expressions in the p-variables, of maximum degree 2. The degree 2 terms in $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ immediately imply

$$V(t, w, z) = V_1(t, w) + V_2(t, z).$$

From $\mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$ it then follows that

$$\frac{\partial V_1(t, w)}{\partial w} = 0 \quad \text{or} \quad \frac{\partial V_2(t, z)}{\partial z} = 0.$$

Let us take $V_2(t, z) = \xi(t)$. But the potential is defined up to an additional function of time, so we can take $\xi(t) = 0$. Then the constant term in $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0$ implies that

$$V_1(t, w) = \beta_1\left(2\int \sigma(t)dt + w\right)\sigma(t) + \beta_2(t)$$

where β_1 and β_2 are arbitrary functions. Let us take

$$V_1(t, w) = \sigma(t)\left(2\int \sigma(t)dt + w\right)^2.$$

The terms linear in p_w, p_z in $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0$ subsequently fix F_1 and F_2,

$$F_1(t) = c_1\sigma(t), \qquad F_2(t) = c_2\sigma(t), \qquad c_1, c_2 \in \mathbb{R}.$$

As we are only looking for a particular solution, take for example $\sigma(t) = t$. Then we have the following Hamiltonian

$$K = t\left(\tfrac{1}{2}(c_1(p_w^2 + p_z^2) + c_2 p_w p_z) + (t^2 + w)^2\right)$$

which is not separable in the w, z-coordinates, but should be in the Darboux-Nijenhuis coordinates associated to R, which is of the form

$$R = \frac{\partial}{\partial w} \otimes dw - \frac{\partial}{\partial z} \otimes dz + 2t\frac{\partial}{\partial w} \otimes dt - 2t\frac{\partial}{\partial z} \otimes dt.$$

Let us finally put our claim to an explicit test. To do so, we first need to determine Darboux-Nijenhuis coordinates (t, Q_1, Q_2) for R. In the Darboux-Nijenhuis coordinates R should be diagonal, but the matrix (R^i_j) is already diagonal. Hence the transformation to the Q-coordinates must preserve this and can be taken to be $w = w(t, Q_1)$, $z = z(t, Q_2)$. To make sure that in the Darboux-Nijenhuis coordinates, the R^i_0-components are zero, the transformation formulas should satisfy

$$\begin{aligned} 2t + \frac{\partial w(t, Q_1)}{\partial t} &= 0 \\ -2t - \frac{\partial z(t, Q_2)}{\partial t} &= 0. \end{aligned}$$

A possible (linear) transformation to Darboux-Nijenhuis coordinates is

$$Q_1 = w - t^2, \qquad Q_2 = z - t^2.$$

The induced change of momenta is then $P_1 = p_w$, $P_2 = p_z$. The Hamiltonian $\widetilde{K}(t, Q, P)$ of the transformed system is

$$\widetilde{K}(t, Q, P) = t\left(\tfrac{1}{2}c_1(P_1^2 + P_2^2) + 2c_2 P_1 P_2 + 2(P_1 + P_2) + Q_1^2\right)$$

and R is given by

$$R = \frac{\partial}{\partial Q_1} \otimes dQ_1 - \frac{\partial}{\partial Q_2} \otimes dQ_2.$$

Since $\partial \widetilde{K}/\partial Q_2 = 0$, the only condition of Forbat which still should be checked is

$$\frac{\partial \widetilde{K}}{\partial P_1}\frac{\partial^2 \widetilde{K}}{\partial Q_1\, \partial t} = \frac{\partial \widetilde{K}}{\partial Q_1}\frac{\partial^2 \widetilde{K}}{\partial P_1\, \partial t}.$$

But this is clearly satisfied. So the Hamilton-Jacobi equation for $\widetilde{K}$ can be solved by separation of variables.

3.5.3 Case 2

First take $k = 1$ and $\rho(t) = c = 1$, $f(t) = 0$ and again $h_1(t, w) = w$ and $h_2(t, z) = z$. As in the previous case, the introduction of w and z suggests the time-dependent coordinate transformation $(t, q_1, q_2) \to (t, w, z)$, with $w = q_2 + \sigma(t) q_1$, $z = q_2 - \sigma(t) q_1$, on E. In the new coordinates (t, w, z), R is given by

$$\begin{aligned} R = (a+b)\frac{\partial}{\partial w} \otimes dw + (a-b)\frac{\partial}{\partial z} \otimes dz + \left(2w - bw\frac{\dot{\sigma}(t)}{\sigma(t)}\right)\frac{\partial}{\partial w} \otimes dt \\ + \left(z + bz\frac{\dot{\sigma}(t)}{\sigma(t)}\right)\frac{\partial}{\partial z} \otimes dt. \end{aligned} \tag{3.65}$$

The induced time-dependent canonical transformation on $J^1\tau^*$ is the same as in case 1b, so also the Hamiltonian K in the new coordinates will be the same (3.62). The condition $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ gives rise to a polynomial expression of degree 2 in the p-variables; the coefficients are themselves polynomials in w and z, with a degree which varies from 0 to 4. It turns out that the degree 2 terms in the p-variables generate, among other conditions, algebraic relations of the form

$$\begin{aligned} A_3(t)F_1(t)F_2(t) = 0, \qquad & A_4(t)F_1(t)F_2(t) = 0, \\ A_3(t)(F_1(t)^2 + F_2(t)^2) = 0, \qquad & A_4(t)(F_1(t)^2 + F_2(t)^2) = 0. \end{aligned}$$

We don't want $F_1(t)$ and $F_2(t)$ both to be zero, so this implies that

$$A_3(t) = 0, \qquad A_4(t) = 0.$$

The coefficient of w^2z^2 then is $A_1(t)A_2(t)F_2(t)$. Since we only want to find a particular solution, let us for example take $F_2(t) = 0$. In this case the coefficient of z^2p_w reduces to $A_2(t)F_1(t)C_2(t) = 0$ and the coefficient of w^2p_z to $A_1(t)F_1(t)C_2(t) = 0$. Let us consider the case $C_2(t) = 0$. The other coefficients of p_w generate first-order differential equations for $A_1(t)$ and $A_5(t)$. We find

$$A_1(t) = \alpha_1\, F_1(t)\, \sigma(t)^{-\frac{5b}{a+b}}\, e^{\frac{10t}{a+b}}, \quad A_5(t) = \alpha_5\, F_1(t)\, \sigma(t)^{-\frac{4b}{a+b}}\, e^{\frac{8t}{a+b}},$$

where α_1, α_5 are arbitrary constants. Similarly, the remaining coefficients of p_z generate first-order differential equations for $A_2(t)$ and $A_6(t)$, we have

$$A_2(t) = \alpha_2\, F_1(t)\, \sigma(t)^{\frac{5b}{a-b}}\, e^{\frac{10t}{a-b}}, \quad A_6(t) = \alpha_6\, F_1(t)\, \sigma(t)^{\frac{4b}{a-b}}\, e^{\frac{8t}{a-b}},$$

where α_2, α_6 are arbitrary constants again. The remaining coefficient of p_w^2 can be seen as a first-order differential equation for $C_1(t)$, with solution

$$C_1(t) = c_1\, F_1(t)\, \sigma(t)^{-\frac{2b}{a+b}}\, e^{\frac{4t}{a+b}} + \frac{b\dot{\sigma}(t) - 2\sigma(t)}{(a+b)\sigma(t)}, \qquad c_1 \in \mathbb{R}.$$

Then the remaining coefficient of p_z^2 can be seen as a first-order differential equation for $F_1(t)$, with solution

$$F_1(t) = \frac{-2b(a\dot{\sigma}(t) + 2\sigma(t))}{c_1\,\sigma(t)^{\frac{a+b}{a-b}}\,e^{\frac{4t}{a-b}}\left(\sigma(t)^{\frac{-4ab}{a^2-b^2}}e^{\frac{-8tb}{a^2-b^2}}(a^2-b^2) - \frac{2bc_2}{a-b}\right)}, \qquad c_2 \in \mathbb{R}.$$

To guarantee that $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$, the only function left to determine is $A_7(t)$ and it follows from the coefficient of $p_w p_z$ that $A_7(t) = 0$. The other two requirements

$$\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0 \quad \text{and} \quad \mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$$

are, in this case, again identically satisfied as a result of the conclusions we drew from the first condition. We thus arrive at the following class of separable Hamiltonians,

$$\begin{aligned} K = {} & \tfrac{1}{2}F_1(t)(p_w^2 + p_z^2) + C_1(t)(wp_w + zp_z) \\ & + A_1(t)w^3 + A_2(t)z^3 + A_5(t)w^2 + A_6(t)z^2. \end{aligned} \tag{3.66}$$

Darboux-Nijenhuis coordinates can be determined in a similar way as in case 1b. A possible transformation to Darboux-Nijenhuis coordinates is

$$Q_1 = \sigma(t)^{\frac{-b}{a+b}}\,e^{\frac{2t}{a+b}}w, \quad Q_2 = \sigma(t)^{\frac{b}{a-b}}\,e^{\frac{2t}{a-b}}z.$$

The induced change of momenta can be written in the form

$$P_1 = \sigma(t)^{\frac{b}{a+b}}\,e^{\frac{-2t}{a+b}}p_w, \quad P_2 = \sigma(t)^{\frac{-b}{a-b}}\,e^{\frac{-2t}{a-b}}p_z.$$

We subsequently find that the Hamiltonian of the transformed system is given by

$$\begin{aligned} \widetilde{K} = {} & F_1(t)\sigma(t)^{\frac{-2b}{a+b}}e^{\frac{4t}{a+b}}\left(\frac{1}{2}P_1^2 + \alpha_1 Q_1^3 + \alpha_5 Q_1^2 + c_1(Q_1P_1 + Q_2P_2)\right) \\ & + F_1(t)\sigma(t)^{\frac{2b}{a-b}}e^{\frac{4t}{a-b}}\left(\frac{1}{2}P_2^2 + \alpha_2 Q_2^3 + \alpha_6 Q_2^2\right) \\ & + \left(\frac{2ab\dot{\sigma}(t)}{(a^2-b^2)\sigma(t)} + \frac{4b}{a^2-b^2}\right)P_2Q_2 \end{aligned} \tag{3.67}$$

and R is given by

$$R = (a+b)\frac{\partial}{\partial Q_1} \otimes dQ_1 + (a-b)\frac{\partial}{\partial Q_2} \otimes dQ_2. \tag{3.68}$$

It can be checked that $\widetilde{K}$ indeed satisfies the Forbat conditions such that the associated Hamilton-Jacobi equation can be solved by separation of variables.

3.5.4 Case 3

For simplicity, take $a = 1$, $k_1 = 1$ and $b_1 = b_2 = 0$, such that R is of the form

$$
\begin{aligned}
R = \rho(t) q_1 \left(\frac{\partial}{\partial q_1} \otimes dq_1 + \frac{\partial}{\partial q_2} \otimes dq_2 \right) + \sigma_1(t) q_2 \frac{\partial}{\partial q_1} \otimes dq_2 + \rho(t) q_2 \frac{\partial}{\partial q_2} \otimes dq_1 \\
+ \left(\dot\rho(t) q_1^2 + \frac{\sigma_1(t)\dot\rho(t) + \rho(t)\dot\sigma_1(t)}{2\rho(t)} q_2^2 \right) \frac{\partial}{\partial q_1} \otimes dt \\
+ \frac{3\sigma_1(t)\dot\rho(t) + \rho(t)\dot\sigma_1(t)}{2\sigma_1(t)} q_1 q_2 \frac{\partial}{\partial q_2} \otimes dt.
\end{aligned}
\tag{3.69}
$$

The eigenfunctions of the matrix R^i_j are given by $\lambda_i = \rho(t) q_1 \pm \sqrt{\sigma_1(t)\rho(t)}\, q_2$. As we learn from the general expression (2.50), the coefficients of a suitable R in Darboux-Nijenhuis coordinates can be arbitrary functions $\lambda_i(Q^i)$, each depending on a single coordinate Q^i. Hence, they can be chosen to be the Q^i themselves. The structure of the R^i_j in (3.69) therefore already tells us what the linear change of coordinates will be after pinning down the freedom in (3.22) and (3.69). The ensuing coordinate change will of course be valid in the domain where $\sigma_1(t)\rho(t)$ is positive.

The condition $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ gives rise to a polynomial expression of degree 2 in the p_i; the coefficients are themselves polynomials in the q^i, with a degree which varies from 2 to 6. It turns out that the p_1^2-terms generate, among other conditions, a first-order differential equation for $c_1(t)$ and algebraic relations which fix $a_2(t)$ in terms of $a_3(t)$ and $a_4(t)$ in terms of $a_1(t)$. Explicitly, we find that

$$
a_2(t) = \frac{\sigma_1(t) a_3(t)}{3\rho(t)}, \qquad a_4(t) = \frac{3\sigma_1(t) a_1(t)}{\rho(t)}, \qquad c_1(t) = C_1 f_2(t)\rho(t)\sigma_1(t) - \frac{\dot\rho(t)}{\rho(t)}
$$

where C_1 is a constant. The remaining coefficient of p_1^2 then determines $a_6(t)$,

$$
\begin{aligned}
a_6(t) = -C_1^2 \sigma_1(t)^2 \rho(t)^2 f_2(t) + C_1 \sigma_1(t)\rho(t) c_2(t) + \tfrac{1}{2} C_1 \sigma_1(t)\dot\rho(t) \\
+ \tfrac{1}{2} C_1 \rho(t)\dot\sigma_1(t) + \frac{\sigma_1(t) a_5(t)}{\rho(t)}.
\end{aligned}
$$

The coefficient of $q_1^2 p_2^2$ is a first-order differential equation for $c_2(t)$ with solution

$$
c_2(t) = C_2 f_2(t)\sigma_1(t)\rho(t) - \frac{1}{2}\left(\frac{\dot\rho(t)}{\rho(t)} + \frac{\dot\sigma_1(t)}{\sigma_1(t)} \right), \qquad C_2 \in \mathbb{R}.
$$

The other p_2^2-term, the coefficient of $q_2^2 p_2^2$, then simplifies to the algebraic relation

$$
\rho(t)^4 \sigma_1(t)^4 f_2(t)^2 (C_1 - C_2)^2 (\rho(t) f_1(t) + \sigma_1(t) f_2(t)) = 0. \tag{3.70}
$$

For this case we will give a full overview of the possible separable Hamiltonians. So we have in principle two possible subcases. However it turns out that the case $f_1(t) = -\sigma_1(t)f_2(t)/\rho(t)$ in the end produces subcases (with constant $l_1 = -1$) of the solutions corresponding with the case $C_1 = C_2$, which we are going to present below.

So, let $C_1 = C_2$, further useful info comes from the monomials of degree 3 or 4 in the coefficient of p_1. We can easily integrate equations for $a_1(t)$, $a_3(t)$, $a_5(t)$ and $a_7(t)$, leading to

$$\begin{aligned} a_1(t) &= \alpha_1 f_1(t)\rho(t)^5, & a_3(t) &= \alpha_3 f_1(t)\rho(t)^{9/2}\sigma_1(t)^{1/2}, \\ a_5(t) &= \alpha_5 f_1(t)\rho(t)^4, & a_7(t) &= \alpha_7 f_1(t)\rho(t)^{7/2}\sigma_1(t)^{1/2}, \end{aligned}$$

where α_1, α_3, α_5 and α_7 are arbitrary constants. The monomials in the coefficient of p_2 then generate two subcases, either $f_1(t) = l_1 f_2(t)\sigma_1(t)/\rho(t)$ with l_1 an arbitrary constant, or $\alpha_1 = \alpha_3 = \alpha_7 = 0$ and $\alpha_5 = \frac{1}{2}C_2^2$.

Subcase 1: $f_1(t) = \dfrac{l_1 f_2(t)\sigma_1(t)}{\rho(t)}$

The only other coefficients to be looked at, which are coefficients of the constant term in the polynomial expression in the p_i, form a system of algebraic relations which fix the constants l_1, α_1 and α_5. There are 3 possible solutions. Note that after determining l_1, α_1 and α_5, the remaining conditions $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0$ and $\mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$ are again identically satisfied.

1. For $l_1 = 1$ we have the following class of separable Hamiltonians,

$$\begin{aligned} H = {} & \tfrac{1}{2}\frac{f_2(t)\sigma_1(t)}{\rho(t)}p_1^2 + \tfrac{1}{2}f_2(t)p_2^2 + \left(C_2 f_2(t)\rho(t)\sigma_1(t) - \frac{\dot\rho(t)}{\rho(t)}\right)p_1q_1 \\ & + \left(C_2 f_2(t)\sigma_1(t)\rho(t) - \tfrac{1}{2}\left(\frac{\dot\rho(t)}{\rho(t)} + \frac{\dot\sigma_1(t)}{\sigma_1(t)}\right)\right)p_2q_2 + \alpha_1 f_2(t)\rho(t)^4\sigma_1(t)q_1^3 \\ & + \tfrac{1}{3}\alpha_3 f_2(t)\rho(t)^{5/2}\sigma_1(t)^{5/2}q_2^3 + \alpha_3 f_2(t)\rho(t)^{7/2}\sigma_1(t)^{3/2}q_1^2q_2 \\ & + 3\alpha_1 f_2(t)\rho(t)^3\sigma_1(t)^2q_1q_2^2 + \alpha_5 f_2(t)\rho(t)^3\sigma_1(t)q_1^2 + \alpha_5 f_2(t)\rho(t)^2\sigma_1(t)^2q_2^2 \\ & + \alpha_7 f_2(t)\rho(t)^{5/2}\sigma_1(t)^{3/2}q_1q_2. \end{aligned} \tag{3.71}$$

In Darboux-Nijenhuis coordinates, the Hamiltonian is given by

$$K = f_2(t)\rho(t)\sigma_1(t)\left(P_1^2 + P_2^2 + C_2(Q_1P_1 + Q_2P_2) + \frac{3\alpha_1 + \alpha_3}{6}Q_1^3 \right.$$
$$\left. + \frac{3\alpha_1 - \alpha_3}{6}Q_2^3 + \frac{2\alpha_5 + \alpha_7}{4}Q_1^2 + \frac{2\alpha_5 - \alpha_7}{4}Q_2^2\right). \tag{3.72}$$

For $\rho(t) = t$, $\sigma_1(t) = 1$, $f_2(t) = 1$, $\alpha_3 = \alpha_2$, $\alpha_5 = 2$ and $\alpha_7 = 0$ this is exactly the example we discussed in our paper [61].

2. For $\alpha_1 = \frac{1}{3}\alpha_3$ and $\alpha_5 = \frac{1}{2}(C_2^2 + \alpha_7)$ we arrive at the following class of separable Hamiltonians

$$H = \tfrac{1}{2}\frac{l_1 f_2(t)\sigma_1(t)}{\rho(t)}p_1^2 + \tfrac{1}{2}f_2(t)p_2^2 + \left(C_2 l_1 f_2(t)\rho(t)\sigma_1(t) - \frac{\dot\rho(t)}{\rho(t)}\right)p_1q_1$$
$$+ \left(C_2 f_2(t)\sigma_1(t)\rho(t) - \tfrac{1}{2}\left(\frac{\dot\rho(t)}{\rho(t)} + \frac{\dot\sigma_1(t)}{\sigma_1(t)}\right)\right)p_2q_2 + \tfrac{1}{3}\alpha_3 l_1 f_2(t)\rho(t)^4\sigma_1(t)q_1^3$$
$$+ \tfrac{1}{3}\alpha_3 l_1 f_2(t)\rho(t)^{5/2}\sigma_1(t)^{5/2}q_2^3 + \alpha_3 l_1 f_2(t)\rho(t)^{7/2}\sigma_1(t)^{3/2}q_1^2q_2$$
$$+ \alpha_3 l_1 f_2(t)\rho(t)^3\sigma_1(t)^2q_1q_2^2 + \tfrac{1}{2}(C_2^2 + \alpha_7)l_1 f_2(t)\rho(t)^2\sigma_1(t)^2(q_1^2 + q_2^2)$$
$$+ \alpha_7 l_1 f_2(t)\rho(t)^{5/2}\sigma_1(t)^{3/2}q_1q_2. \tag{3.73}$$

In Darboux-Nijenhuis coordinates, the Hamiltonian is given by

$$K = f_2(t)\rho(t)\sigma_1(t)\left(\frac{l_1+1}{2}(P_1^2 + P_2^2) + \frac{l_1-1}{2}P_1P_2 + C_2\frac{l_1+1}{2}(Q_1P_1 + Q_2P_2)\right.$$
$$+ C_2\frac{l_1-1}{2}(Q_2P_1 + Q_1P_2) + \tfrac{1}{3}\alpha_3 l_1 Q_1^3 + \frac{C_2^2(l_1+1) + 4\alpha_7 l_1}{8}Q_1^2$$
$$\left. + C_2^2\frac{l_1+1}{8}Q_2^2 + C_2^2\frac{l_1-1}{4}Q_1Q_2\right). \tag{3.74}$$

3. Finally, for $\alpha_1 = -\frac{1}{3}\alpha_3$ and $\alpha_5 = \frac{1}{2}(C_2^2 - \alpha_7)$ we have a class of separable

Hamiltonians of the form

$$\begin{aligned}
H &= \tfrac{1}{2}\frac{l_1 f_2(t)\sigma_1(t)}{\rho(t)}p_1^2 + \tfrac{1}{2}f_2(t)p_2^2 + \left(C_2 l_1 f_2(t)\rho(t)\sigma_1(t) - \frac{\dot\rho(t)}{\rho(t)}\right)p_1q_1 \\
&+ \left(C_2 f_2(t)\sigma_1(t)\rho(t) - \tfrac{1}{2}\left(\frac{\dot\rho(t)}{\rho(t)} + \frac{\dot\sigma_1(t)}{\sigma_1(t)}\right)\right)p_2q_2 - \tfrac{1}{3}\alpha_3 l_1 f_2(t)\rho(t)^4\sigma_1(t)q_1^3 \\
&+ \tfrac{1}{3}\alpha_3 l_1 f_2(t)\rho(t)^{5/2}\sigma_1(t)^{5/2}q_2^3 + \alpha_3 l_1 f_2(t)\rho(t)^{7/2}\sigma_1(t)^{3/2}q_1^2q_2 \\
&- \alpha_3 l_1 f_2(t)\rho(t)^3\sigma_1(t)^2 q_1q_2^2 + \tfrac{1}{2}(C_2^2-\alpha_7)l_1 f_2(t)\rho(t)^2\sigma_1(t)^2(q_1^2+q_2^2) \\
&+ \alpha_7 l_1 f_2(t)\rho(t)^{5/2}\sigma_1(t)^{3/2}q_1q_2. \qquad (3.75)
\end{aligned}$$

In Darboux-Nijenhuis coordinates, the Hamiltonian is given by

$$\begin{aligned}
K = f_2(t)\rho(t)\sigma_1(t)&\left(\frac{l_1+1}{2}(P_1^2+P_2^2) + \frac{l_1-1}{2}P_1P_2 + C_2\frac{l_1+1}{2}(Q_1P_1+Q_2P_2)\right. \\
&+C_2\frac{l_1-1}{2}(Q_2P_1+Q_1P_2) - \tfrac{1}{3}\alpha_3 l_1 Q_2^3 + C_2^2\frac{(l_1+1)}{8}Q_1^2 \\
&\left.+\frac{C_2^2(l_1+1)-4\alpha_7 l_1}{8}Q_2^2 + C_2^2\frac{l_1-1}{4}Q_1Q_2\right). \qquad (3.76)
\end{aligned}$$

Subcase 2: $\alpha_1 = \alpha_3 = \alpha_7 = 0$ **and** $\alpha_5 = \frac{1}{2}C_2^2$

All the remaining coefficients of $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}(X_h)) = 0$ are then automatically zero and also $\mathcal{L}_{X_h}\omega_R(X_h, \widetilde{R}^2(X_h)) = 0$ and $\mathcal{L}_{X_h}\omega_R(\widetilde{R}(X_h), \widetilde{R}^2(X_h)) = 0$ are satisfied. So we arrive at the following class of separable Hamiltonians

$$\begin{aligned}
H &= \tfrac{1}{2}f_1(t)p_1^2 + \tfrac{1}{2}f_2(t)p_2^2 + \left(f_1(t)\rho(t)^2 - \frac{\dot\rho(t)}{\rho(t)}\right)p_1q_1 \\
&+ \left(C_2 f_2(t)\sigma_1(t)\rho(t) - \tfrac{1}{2}\left(\frac{\dot\rho(t)}{\rho(t)} + \frac{\dot\sigma_1(t)}{\sigma_1(t)}\right)\right)p_2q_2 \\
&+ \tfrac{1}{2}C_2^2 f_1(t)\rho(t)^4 q_1^2 - \tfrac{1}{2}f_2(t)\rho(t)^2\sigma_1(t)^2q_2^2. \qquad (3.77)
\end{aligned}$$

In Darboux-Nijenhuis coordinates, the Hamiltonian is given by

$$\begin{aligned}
K = \rho(t)&\left[\tfrac{1}{2}(f_1(t)\rho(t)+\sigma_1(t)f_2(t))(P_1^2+P_2^2+C_2(Q_1P_1+Q_2P_2)+\tfrac{1}{4}C_2^2(Q_1^2+Q_2^2))\right. \\
&\left.+(f_1(t)\rho(t)-\sigma_1(t)f_2(t))(P_1P_2+\tfrac{1}{2}C_2(Q_2P_1+Q_1P_2)+\tfrac{1}{4}C_2^2Q_1Q_2)\right]. \qquad (3.78)
\end{aligned}$$

CHAPTER

4

DRIVEN COFACTOR SYSTEMS

'Driven cofactor systems' are partially decoupling second-order differential equations of a special kind. They were introduced by Lundmark and Rauch-Wojciechowski [41] in Euclidean space. In this chapter we will give an intrinsic, geometrical characterization of such systems, and explain the basic underlying concepts based on [54]. In the sequel we discuss the more intricate part of the theory. It involves in the first place understanding all details of an algorithmic construction of quadratic first integrals and their involutivity. It secondly requires explaining the subtle way in which suitably constructed canonical transformations reduce the Hamilton-Jacobi problem of the (a priori time-dependent) driven part of the system into that of an equivalent autonomous system of Stäckel type.

4.1 Cofactor systems

Before studying in detail driven cofactor systems we first introduce the notion of a cofactor system in Euclidean space and subsequently consider cofactor systems on Riemannian manifolds.

4.1.1 Cofactor systems in Euclidean space

To the best of our knowledge, the idea of a cofactor system stems from a paper on some particular Newtonian systems in Euclidean space which is mainly restricted to systems with two degrees of freedom [51]. But we will refer most often to a paper of Lundmark [40], which contains the most general exposition of the theory, in particular the generalization to systems of arbitrary dimension.

We first recall the notion of an elliptic coordinates matrix.

Definition 4.1. *An elliptic coordinates matrix in $\mathbb{R}^n$ is a symmetric $n \times n$-matrix $G(q)$ whose entries are quadratic polynomials in the q^i's $(i = 1, \dots, n)$ of the form*

$$G_{ij}(q) = \alpha q^i q^j + \beta_i q^j + \beta_j q^i + \gamma_{ij}, \quad \alpha, \beta_i, \gamma_{ij} = \gamma_{ji} \in \mathbb{R}$$

or, in matrix notation (T denotes the transpose of a matrix),

$$G(q) = \alpha q q^T + q\beta^T + \beta q^T + \gamma, \quad \textit{where} \quad \alpha \in \mathbb{R}, \beta \in \mathbb{R}^n, \gamma = \gamma^T \in \mathbb{R}^{n\times n}.$$

For clarity, elements q in $\mathbb{R}^n$ are considered as column vectors, so qq^T is a $n \times n$-matrix. Note that an elliptic coordinates matrix is in fact a special conformal Killing tensor (Definition 1.28) defined in Euclidean space.

Originally, Lundmark defined a cofactor system in Euclidean space as follows.

Definition 4.2. *A cofactor system in Euclidean space is a mechanical system of the form*

$$\ddot{q} = -A(q)^{-1}\nabla W(q),$$

with $A = \operatorname{cof} G(q)$ the cofactor matrix of a nonsingular elliptic coordinates matrix $G(q)$ and where W is a function on $\mathbb{R}^n$ and ∇W denotes its gradient. It can be shown that the function $E = \frac{1}{2}\dot{q}^T A(q)\dot{q} + W(q)$ is a first integral, which is called a quadratic integral of cofactor type.

Recall that the cofactor tensor A of a type (1,1) tensor field J (notation $A = \operatorname{cof} J$) is defined by the relation $JA = AJ = (\det J)I$.

The term 'quadratic integral of cofactor type' refers to the fact that the matrix of the quadratic part of the first integral comes from the cofactor tensor of a tensor which has special properties with respect to the (Euclidean) metric. The other point to be emphasized is that the Newtonian systems under consideration have force terms which are of nonconservative type, albeit of a very special nature, determined by a scalar function $W(q)$ and also by the cofactor tensor $A(q)$.

4.1.2 Cofactor systems on Riemannian manifolds

It was recognized in that the work of Lundmark could easily be generalized to systems with a kinetic energy associated to an arbitrary Riemannian metric. As was pointed out in [19], the results of this section are even valid for general pseudo-Riemannian manifolds. Nevertheless, we will only talk about the Riemannian case, mainly for the treatment that follows. Consider $T = \frac{1}{2}g_{\alpha\beta}(q)v^\alpha v^\beta$, a kinetic energy function on the tangent bundle TM of a Riemannian manifold M with metric g. The mechanical systems we will be talking about belong to the class of nonconservative Lagrangian systems, governed by equations of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} = Q_\alpha. \tag{4.1}$$

The nonconservative forces Q_α are assumed to depend on the position variables only and thus can be viewed as components of a 1-form $\mu = Q_\alpha(q)dq^\alpha$ on M. To introduce the notion of a cofactor system on a general Riemannian manifold as in [19], we first need to recall some general concepts.

Since an elliptic coordinates matrix is a special conformal Killing tensor defined in Euclidean space, it should not be a surprise that a scKt is involved in the definition of a cofactor system on a Riemannian manifold. So, consider a nonsingular scKt J on M (Definition 1.28),

$$J_{\alpha\beta|\gamma} = \tfrac{1}{2}(\alpha_\alpha g_{\beta\gamma} + \alpha_\beta g_{\alpha\gamma}), \quad \text{which further implies that } \alpha = \alpha_\beta dq^\beta = d(\operatorname{tr} J). \tag{4.2}$$

Making use of J, we can define two differential operators. In the first place we can consider d_J as in Definition 1.11. From Theorem 1.30 we know that the Nijenhuis torsion of the scKt J vanishes and this implies that ${d_J}^2 = 0$. Therefore (d, d_J) is a simple bi-differential calculus (see Definition 1.21). Moreover, since J is nonsingular, d_J satisfies also a sort of Poincaré lemma, namely $d_J\theta = 0$ for a k-form θ on M if and only if there exists locally a $(k-1)$-form ψ on M such that $\theta = d_J\psi$ [63].
Secondly, we can define a differential operator D_J as

$$D_J\rho = d_J\rho + d(\operatorname{tr} J)\wedge\rho = (\det J)^{-1}d_J((\det J)\rho) \quad \text{for all differential forms } \rho \text{ on } M. \tag{4.3}$$

The equality of both expressions in the defining relation of D_J follows from Lemma 1.17. Remark that D_J is not a derivation in the sense of the Frölicher and Nijenhuis theory. But (d, D_J) is a gauged bi-differential calculus (see Definition 1.24) since D_J satisfies ${D_J}^2 = 0$. Moreover D_J has the property 'D_J-closed' is equivalent to 'locally D_J-exact', since d_J satisfies a Poincaré lemma.

We are now ready to formulate the definition of a cofactor system on a Riemannian manifold.

Definition 4.3. *A cofactor system is a triple (g, μ, J) on a Riemannian manifold M where g is the metric, μ is a 1-form on M and J is a nonsingular special conformal Killing tensor on M such that $D_J\mu = 0$.*

As we will illustrate now, cofactor systems can be interpreted in a natural way as nonconservative systems admitting a *quasi-Hamiltonian representation*.

Suppose that we have a type (1,1) tensor field J on M with vanishing Nijenhuis torsion. Then J induces a Poisson structure $P_J = \tilde{J} \circ P_0$, on T^*M, where $\tilde{J}$ is the complete lift of J to T^*M (see (2.12)) and P_0 is the standard Poisson map[2] on T^*M,

$$P_0 = \frac{\partial}{\partial q^\alpha} \wedge \frac{\partial}{\partial p_\alpha}.$$

The coordinate expression of P_J is given by

$$P_J = J^\alpha_\beta \frac{\partial}{\partial q^\alpha} \wedge \frac{\partial}{\partial p_\beta} - \frac{1}{2} p_\gamma \left(\frac{\partial J^\gamma_\alpha}{\partial q^\beta} - \frac{\partial J^\gamma_\beta}{\partial q^\alpha} \right) \frac{\partial}{\partial p_\alpha} \wedge \frac{\partial}{\partial p_\beta}. \tag{4.4}$$

Consider a SODE Γ associated with a nonconservative system defined by a metric tensor field g and 1-form $\mu = Q_\alpha dq^\alpha$ on M,

$$\Gamma = v^\alpha \frac{\partial}{\partial q^\alpha} - \Gamma^\alpha_{\beta\gamma} v^\beta v^\gamma \frac{\partial}{\partial v^\alpha} + Q^\alpha \frac{\partial}{\partial v^\alpha},$$

where $Q^\alpha = g^{\alpha\beta} Q_\beta$, and denote by $\hat{\Gamma} \in \mathcal{X}(T^*M)$ the image of the given SODE Γ under the Legendre map,

$$\hat{\Gamma} = g^{\alpha\beta} p_\beta \frac{\partial}{\partial q^\alpha} + \Gamma^\gamma_{\mu\alpha} g^{\mu\delta} p_\gamma p_\delta \frac{\partial}{\partial p_\alpha} + Q_\alpha \frac{\partial}{\partial p_\alpha}. \tag{4.5}$$

The question now is under what circumstances one can find a J with $N_J = 0$ such that the given system satisfies

$$F\hat{\Gamma} = -P_J(dH) \tag{4.6}$$

[2]Note that the sign convention we adopt here is in accordance with Section 1.1, but differs from the sign convention in [19], [54] and [55].

for some functions F and $H = \frac{1}{2}A^{\alpha\beta}p_\alpha p_\beta + V(q)$ with A a symmetric tensor and V a function on M. In [19] it was shown that this is possible under the conditions that J is a scKt with cofactor tensor A and μ satisfies $A\mu = -dV$. The function F is then given by $F = \det J$. The condition on μ is locally equivalent with $D_J\mu = 0$ since 'D_J-closed' is equivalent to 'locally D_J-exact'. Since a special conformal Killing tensor automatically has zero Nijenhuis torsion (see Proposition 1.30), the result can be summarized as follows.

Theorem 4.4. *A nonconservative system Γ on TM, generated by a metric tensor field g and 1-form μ on M, has a quasi-Hamiltonian representation $F\hat{\Gamma} = -P_J(dH)$, where J is a $(1,1)$ tensor field on M and H is a function on T^*M quadratic in the momenta, if and only if it is a cofactor system (g, μ, J).*

Note that the function H is a *quadratic first integral of cofactor type*, which explains the term 'cofactor system'.

Cofactor pair systems constitute an interesting subclass: the nonconservative system then has a double cofactor representation.

Definition 4.5. *A cofactor pair system is a nonconservative system Γ defined by (g, μ) where g is the metric on a Riemannian manifold, μ is a 1-form which satisfies $D_J\mu = D_L\mu = 0$ for two nonsingular special conformal Killing tensors J and L.*

So two modified Poisson tensors P_J and P_L can be defined.

Theorem 4.6. *[19] A cofactor pair system is completely integrable. It admits n integrals of motion*

$$H_{(m)} = \frac{1}{2}A^{\alpha\beta}_{(m)}p_\alpha p_\beta + V_{(m)}(q),$$

which are in involution with respect to the Poisson brackets associated with P_J and P_L.

In Section 4.3 we will discuss the construction of these first integrals in detail for the specific case of a driven cofactor system.

4.2 Driven cofactor systems: definition

As the title of this chapter reflects, there is an extra aspect about cofactor systems we want to study. So called 'driven cofactor systems' were introduced in [41], still in the context of mechanical systems with Euclidean kinetic energy metric and, hence, having no terms quadratic in the velocities in the second-order equations of motion. Briefly, the systems discussed in [41] are of the form

$$\begin{aligned} \ddot{y}^i &= Q^i(y^j), \qquad & i &= 1, \ldots, m, \\ \ddot{x}^a &= Q^a(y^j, x^b), \qquad & a &= 1, \ldots, n. \end{aligned}$$

They exhibit a given partial decoupling whereby the decoupled y-system is referred to as the *driving system* and the remaining x-equations as the *driven system*. In addition, it is assumed that the overall system is of cofactor type and that, more restrictively, the force terms Q^a come from a potential, parametrically depending on the driving y-coordinates, such that the driven system has a standard Hamiltonian representation. These are rather strong conditions, but they were shown to lead to quite striking conclusions. First of all the driving system turns out to be of cofactor type in its own right. Secondly, the driven system, when regarded as a time-dependent system along solutions $y(t)$ of the driving system, has n (time-dependent) quadratic first integrals. Most astonishingly, however, the authors managed to show that (under some technical assumptions) there exists a time-dependent canonical transformation, which has the effect of shifting the time-dependence in the Hamiltonian of the driven system to an overall factor, so that an autonomous Hamiltonian can be identified which turns out to be of Stäckel type. The idea is to extend in the following sections these results to general driven cofactor systems in such a way that they can be understood in more intrinsic terms.

In order to define driven cofactor systems on a Riemannian manifold, we first introduce an intrinsic characterization of systems which exhibit partial decoupling.

4.2.1 Submersive systems

Second-order differential equations for which one can find suitable coordinates in which the equations partially decouple are called *submersive* in [36]. In this paper, the first intrinsic characterization of submersiveness was given in terms of properties which can be verified prior to the construction of decoupling coordinates. For that purpose the authors used tangent bundle geometry techniques. But in [54], and even before in [47], it was argued that the tools in the context of the calculus along the tangent bundle projection are also well suited for the intrinsic study of submersive systems, so we will follow that approach. One of the main ingredients is the notion of a distribution along the projection $\tau : TM \to M$ and the conditions for the integrability of such a distribution. Therefore we will first briefly recall these concepts. For more details and proofs we refer to [47]. For the basic definitions and notations in the context of the calculus along the tangent bundle projection we refer to Section 1.2.2.

Definition 4.7. *An r-dimensional distribution $\mathcal{D}$ along τ is a smooth choice of an r-dimensional subspace $\mathcal{D}(v)$ of $T_{\tau(v)}M$ for every $v \in TM$. We say that a vector field X along τ belongs to $\mathcal{D}$, if $X(v) \in \mathcal{D}(v)$ for each $v \in TM$.*

By a smooth choice, we mean that in a neighborhood of each $v \in TM$ there exist r independent vector fields along τ which span $\mathcal{D}$ in that neighbourhood.

Interesting additions to the previous definition, are the following.

Definition 4.8. *A distribution $\mathcal{D}$ along τ is said to be basic if there exists a distribution $\mathcal{E}$ on M such that $\mathcal{D}(v) = \mathcal{E}(\tau(v))$ for each $v \in TM$. $\mathcal{D}$ is called involutive if it is basic and if $\mathcal{E}$ is involutive. An integral submanifold of $\mathcal{E}$ is said to be an integral submanifold of $\mathcal{D}$.*

It is immediately clear that $\mathcal{D}$ is basic if and only if it is locally generated over $C^\infty(TM)$ by vector fields on M. In this case, i.e. when $\mathcal{D} = \mathrm{sp}\{Z^1, \ldots, Z^r\}$ where the Z^i are basic vector fields, every vector field Z along τ in $\mathcal{D}$ can be written as $Z = \sum_{i=1}^r \rho_i Z^i$, $\rho_i \in C^\infty(TM)$. Then

$$\mathrm{D}^{\mathrm{V}}_X Z = \sum_{i=1}^{r} (\mathrm{D}^{\mathrm{V}}_X \rho_i) Z^i, \qquad \forall X \in \mathcal{X}(\tau).$$

Thus $\mathrm{D}^{\mathrm{V}}_X Z$ will belong to $\mathcal{D}$ for every $Z \in \mathcal{D}$, meaning that if $\mathcal{D}$ is basic, it is D^{V}-invariant. One can prove that the converse is also true, so we have the following proposition.

Proposition 4.9. *A distribution along τ is basic if and only if it is D^{V}-invariant.*

Making use of the horizontal covariant derivative (1.7), one can define the *horizontal bracket* $[.,.]_H$ of two vector fields along τ. It's given by

$$[X,Y]_H = \mathrm{D}^H_X Y - \mathrm{D}^H_Y X, \qquad \forall X,Y \in \mathcal{X}(\tau). \tag{4.7}$$

It is easy to see that the horizontal bracket of two basic vector fields coincides with the ordinary Lie bracket of such vector fields on M. So, a basic distribution is involutive if and only if it is closed under the horizontal bracket. Hence, a distribution along τ is involutive if and only if it is D^V-invariant and closed under the horizontal bracket.

So, coming back to the main topic of this section, the question is, given a SODE Γ on TM, when does there exist a foliation of M so that in a neighbourhood of each point of M there exist adapted local coordinates (y^i, x^a) such that the given SODE partially decouples into equations of the form

$$\begin{aligned} \ddot{y}^i &= f^i(y,\dot{y}), && i = 1,\dots,m, \\ \ddot{x}^a &= f^a(x,y,\dot{x},\dot{y}), && a = 1,\dots,n \ \ (\text{here } n+m = \dim M). \end{aligned}$$

Proposition 4.10. *A* SODE Γ *on* TM *is submersive if and only if there exists a distribution* K *along* $\tau : TM \to M$*, such that*

$$\Phi(K) \subset K, \qquad \nabla K \subset K, \qquad \mathrm{D}^V_Z K \subset K, \qquad \forall Z \in \mathcal{X}(\tau). \tag{4.8}$$

Proof. From the D^V-invariance it follows that K is basic and thus generated by basic vector fields. The D^V- and ∇-invariance together with the commutator property (easy to verify in coordinates)

$$[\nabla, \mathrm{D}^V_X]Y = \mathrm{D}^V_{\nabla X} Y - \mathrm{D}^H_X Y, \qquad \forall X,Y \in \mathcal{X}(\tau),$$

implies D^H-invariance. From (4.7) it then follows that K is closed under the horizontal bracket and thus involutive. As the horizontal bracket of basic vector fields reduces to the usual Lie bracket, the result is that K is generated by an integrable distribution. Let us introduce adapted coordinates (y^i, x^a), x^a coordinates on the integral submanifolds of K and y^i transversal coordinates, K is then generated by the basic vector fields $\partial/\partial x^a$. We have that

$$\nabla\left(\frac{\partial}{\partial x^a}\right) = -\frac{1}{2}\frac{\partial f^i}{\partial \dot{x}^a}\frac{\partial}{\partial y^i} - \frac{1}{2}\frac{\partial f^b}{\partial \dot{x}^a}\frac{\partial}{\partial x^b}.$$

Then the ∇-invariance of K implies that the forces f^i do not depend on $\dot{x}^a$ and hence $\Gamma^i_a = \partial f^i/\partial\dot{x}^a = 0$. The coordinate expression of Φ^i_a is then $-\partial f^i/\partial x^a$. But K is also invariant under the Jacobi endomorphism, so since

$$\Phi\left(\frac{\partial}{\partial x^a}\right) = \Phi^i_a\frac{\partial}{\partial y^i} + \Phi^b_a\frac{\partial}{\partial x^b},$$

this implies that $\partial f^i/\partial x^a = 0$ as well. Thus Γ is submersive.
Conversely, assume that Γ is submersive, there exists then an integrable distribution $\mathcal{D}$ on TM which is S-regular [36], i.e. $\mathcal{D}$ is regular and generated by vertical and complete lifts of vector fields on the base M. An S-regular distribution on TM can be interpreted as the lifted distribution of a distribution K along τ, if K is basic and involutive. So $K = \mathrm{sp}\{\partial/\partial x^a\}$ and should be D^V-invariant and closed under the horizontal bracket. Since Γ is submersive, $\partial f^i/\partial x^a = \partial f^i/\partial\dot{x}^a = 0$ and it then immediately follows that K is also invariant under the Jacobi endomorphism Φ and the dynamical covariant derivative ∇. □

It is well known that, if a SODE Γ is Lagrangian, the Hessian of this Lagrangian is a symmetric 2-covariant tensor field g along τ satisfying *Helmholtz conditions* [46]:

1. Φ is g-symmetric: $g(\Phi X, Y) = g(X, \Phi Y)$,
2. $\mathrm{D}^V g$ is symmetric: $\mathrm{D}^V_X g(Y,Z) = \mathrm{D}^V_Z g(Y,X)$,
3. $\nabla g = 0$.

We have non-conservative systems in mind, but let us start by investigating what submersiveness means in the presence of a Riemannian metric g on M, satisfying

$$\nabla g = 0, \text{ or equivalently, } g_{ij|k} = 0.$$

Denote by $K^\perp$ the orthogonal complement of K with respect to the Riemannian metric: $g(K, K^\perp) = 0$. For a submersive system it follows from $\nabla g = 0$ and $\mathrm{D}^V_Z g = 0$ that

$$\nabla(K^\perp) \subset K^\perp, \qquad \mathrm{D}^V_Z K^\perp \subset K^\perp, \qquad \forall Z \in \mathcal{X}(\tau).$$

From the g-symmetry of Φ it also follows that $\Phi(K^\perp) \subset K^\perp$. Thus $K^\perp$ also satisfies the conditions of Proposition 4.10. So we can conclude that a submersive Lagrangian system necessarily decouples in two different ways. Moreover $K^\perp$ inherits the properties $\nabla K^\perp \subset K^\perp$ and $\mathrm{D}^V_Z K^\perp \subset K^\perp$ from K, such that $K^\perp$ is also integrable. This is enough to conclude that the two complementary distributions which are both integrable, are in fact simultaneously integrable. Therefore a submersive Lagrangian system actually decouples into two separate systems. In the following, we will show that in the presence of nonconservative forces, this is no longer the case.

4.2.2 Submersive nonconservative systems

By combining the geometric notion of submersiveness and the concept of a cofactor system, the following generalization and coordinate free formulation of a driven cofactor system was derived in [54]. Note that, from now on, we require the metric to be strictly Riemannian (i.e. positive definite).

Definition 4.11. *A driven cofactor system is a cofactor system (g, μ, J), for which there exists a distribution K along the projection $\tau : TM \to M$, with the properties*

$$\Phi(K) \subset K, \quad \nabla K \subset K, \quad \mathrm{D}^V_Z K \subset K, \qquad \forall Z \in \mathcal{X}(\tau), \tag{4.9}$$

$$d\mu(K, K) = 0, \quad \mathrm{D}^H \mu(K^\perp, K) \neq 0. \tag{4.10}$$

We will now clarify the different assumptions in this definition.

As was discussed in Proposition 4.10 the existence of a distribution K along τ having the properties (4.9) precisely means that locally there exist coordinates (y^i, x^a) on M, adapted to the integrable distribution on M which spans K, such that the given SODE partially decouples into equations of the form

$$\ddot{y}^i = f^i(y, \dot{y}), \qquad i = 1, \ldots, m, \tag{4.11}$$

$$\ddot{x}^a = f^a(x, y, \dot{x}, \dot{y}), \qquad a = 1, \ldots, n, \quad (\text{here } n + m = \dim M). \tag{4.12}$$

But there is more to it in this context, which brings us to the final requirements (4.10) in Definition 4.11. Note that we use Greek indices $(\alpha, \beta, \ldots)$ for the coordinates q^α on M and Latin indices (i, a) for the adapted coordinates.

The SODE associated to equations of motion of the form (4.1), is given by

$$\Gamma = \tilde{\Gamma} + Q^\beta \frac{\partial}{\partial v^\beta}, \tag{4.13}$$

where $\tilde{\Gamma}$ is the *geodesic spray* of g,

$$\tilde{\Gamma} = v^\alpha \frac{\partial}{\partial q^\alpha} - \tilde{\Gamma}^\alpha_{\beta\gamma} v^\beta v^\gamma \frac{\partial}{\partial v^\alpha},$$

and $Q^\beta = g^{\beta\alpha} Q_\alpha$, if $\mu = Q_\alpha dq^\alpha$. Comparing the geometrical tools provided by the SODEs Γ and $\widetilde{\Gamma}$, respectively, we observe first of all that

$$\Gamma^\alpha_\beta = \widetilde{\Gamma}^\alpha_\beta = \widetilde{\Gamma}^\alpha_{\beta\gamma} v^\gamma.$$

It follows that $H_\alpha = \widetilde{H}_\alpha$ (see (1.6)), while ∇ and $\widetilde{\nabla}$ (1.8) coincide on basic tensor fields, but obviously differ on functions on TM. We further have,

$$\Phi = \widetilde{\Phi} - \mathrm{D}^{\tilde{H}} Q, \quad \text{where} \quad Q = Q^\alpha \frac{\partial}{\partial q^\alpha}, \quad (\mathrm{D}^{\tilde{H}} Q)^\alpha_\beta = Q^\alpha{}_{|\beta}.$$

Note that the operation $\mathrm{D}^{\tilde{H}}$ is defined on a p-covariant q-contravariant tensor field W along τ by, $\alpha_i \in \mathcal{X}^*(\tau)$ and $X_i \in \mathcal{X}(\tau)$,

$$[\mathrm{D}^{\tilde{H}} W](\alpha_1, \ldots, \alpha_q, X_1, \ldots, X_p) = [\mathrm{D}^{\tilde{H}}_{X_1} W](\alpha_1, \ldots, \alpha_q, X_2, \ldots, X_p).$$

But the geodesic spray $\widetilde{\Gamma}$ is Lagrangian, hence

$$\widetilde{\nabla} g = 0, \quad \text{and} \quad \widetilde{\Phi} \lrcorner\, g \text{ is symmetric.}$$

It follows that $\nabla g = 0$ as well, but of course we insist on $d\mu \neq 0$ to avoid that Γ would also be Lagrangian.

Now, it follows from (4.9) that Γ is submersive. Consider the complement $K^\perp$ of K with respect to g. As before, since $\nabla g = 0$, we have

$$\nabla K^\perp \subset K^\perp \quad \text{and} \quad \mathrm{D}^{\mathrm{v}}_Z K^\perp \subset K^\perp, \qquad \forall Z \in \mathcal{X}(\tau)$$

but generally

$$\begin{aligned} g(\Phi K, K^\perp) &= g(\tilde{\Phi} K, K^\perp) - \mathrm{D}^{H} \mu(K, K^\perp) \\ &= g(K, \Phi K^\perp) + \mathrm{D}^{H} \mu(K^\perp, K) - \mathrm{D}^{H} \mu(K, K^\perp). \end{aligned}$$

The left-hand side is zero, but to avoid splitting of Γ into two separate subsystems, we want that $g(K, \Phi K^\perp) \neq 0$. However, since $\tilde{\Gamma}$ is Lagrangian and also submersive, $g(\tilde{\Phi} K, K^\perp) = 0$. From the first line it then follows that $\mathrm{D}^{H} \mu(K, K^\perp) = 0$. So if we want to avoid that $g(K, \Phi K^\perp)$ is zero, $\mathrm{D}^{H} \mu(K^\perp, K)$ should necessarily be different from zero. So the second assumption in (4.10) guarantees that there will be a partial coupling between the driving and driven part of the dynamics.

Finally, the assumption $d\mu(K,K)=0$ will guarantee, in adapted coordinates, that the $Q_a(y,x)$ satisfy

$$\frac{\partial Q_a}{\partial x^b}-\frac{\partial Q_b}{\partial x^a}=0$$

and thus that the driven part has force terms Q_a which are derivable from a potential energy function (parametrically depending on the driving coordinates).

Remark, as we said before, a result of $\nabla g=0$ is that $K^\perp$ inherits the properties $\nabla K^\perp\subset K^\perp$ and $\mathrm{D}^{\mathrm{V}}_Z K^\perp\subset K^\perp$ from K and this is enough to conclude that the two distributions are in fact simultaneously integrable. Hence we can choose x^a and y^i coordinates which are adapted to K and $K^\perp$ at the same time, i.e. in such a way that

$$K=\mathrm{sp}\left\{\frac{\partial}{\partial x^a}\right\},\qquad K^\perp=\mathrm{sp}\left\{\frac{\partial}{\partial y^i}\right\}.$$

It also follows that the kinetic energy part in the equations of motion (4.1) decouples completely. In other words, if we put $g_1:=g|_{K^\perp}$ and $g_2:=g|_K$, then in adapted coordinates:

$$g_1=g_{ij}(y)dy^i\otimes dy^j,\qquad g_2=g_{ab}(x)dx^a\otimes dx^b, \tag{4.14}$$

while $g_{ia}=g_{ai}=0$. Similarly, for the corresponding connection coefficients, we have

$$\Gamma^i_{jk}=\Gamma^i_{jk}(y),\qquad \Gamma^a_{bc}=\Gamma^a_{bc}(x),$$

and all other connection coefficients are zero. Hence, in coordinates adapted to K and $K^\perp$ the equations of motion will take the form

$$\begin{aligned}\ddot y^i&=-\Gamma^i_{jk}(y)\dot y^j\dot y^k+Q^i(y),\\ \ddot x^a&=-\Gamma^a_{bc}(x)\dot x^b\dot x^c+Q^a(y,x).\end{aligned}$$

4.3 The cofactor pair scheme and $n+1$ quadratic first integrals

In this section we show that a driven cofactor system has a second, in some sense degenerate cofactor representation, so that it is a kind of degenerate cofactor pair system. Then the interesting feature, which follows from Theorem 4.6, is that it gives rise to a family of quadratic first integrals which are in involution with respect to a double Poisson structure. We develop a scheme to construct these first integrals and illustrate the involutiveness.

The distributions K and $K^\perp$ are regular, basic and integrable, so they give rise to two complementary subbundles of TM and corresponding submodules of $\mathcal{X}(\tau)$ which will also be denoted by K and $K^\perp$, respectively. It is then appropriate to introduce complementary projection operators

$$P_1 : \mathcal{X}(M) \to K^\perp, \qquad P_2 : \mathcal{X}(M) \to K.$$

We thus have $P_1 + P_2 = I_N$ (the identity (1,1)-tensor on the $m+n = N$-dimensional manifold M), $P_1 \circ P_2 = P_2 \circ P_1 = 0$, $P_i^2 = P_i$, and we occasionally put $P_1|_{K^\perp} = I_m,\ P_2|_K = I_n$ with in adapted coordinates

$$I_m = \frac{\partial}{\partial y^i} \otimes dy^i, \qquad I_n = \frac{\partial}{\partial x^a} \otimes dx^a.$$

Now, we look at the scKt J (for its action on vector fields) as the sum of the following four parts[3]

$$J_i = P_i \circ J \circ P_i,\ i = 1,2, \quad J_{12} = P_1 \circ J \circ P_2, \quad J_{21} = P_2 \circ J \circ P_1, \tag{4.15}$$

and we shall also use a similar notation for other type (1,1) tensor fields of interest further on. In fact, the matrix representation of J has the following block stucture (for its action on vector fields)

$$(J^\alpha_\beta) = \begin{pmatrix} J_1 & J_{12} \\ J_{21} & J_2 \end{pmatrix}.$$

Recall that the scKt conditions (1.11), when expressed in terms of the coefficients of the original type (1,1) tensor field, take the form

$$J^\alpha_{\beta|\gamma} := \frac{\partial J^\alpha_\beta}{\partial q^\gamma} - J^\alpha_\sigma \Gamma^\sigma_{\beta\gamma} + J^\sigma_\beta \Gamma^\alpha_{\sigma\gamma} = \tfrac{1}{2}(\alpha_\beta \delta^\alpha_\gamma + \alpha_\sigma g^{\sigma\alpha} g_{\beta\gamma}). \tag{4.16}$$

[3] Important notational convention: In principle, such tensor fields act on the whole module of vector fields on M. We shall use the same notation, however, when we consider their restriction to the appropriate submodule K or $K^\perp$ on which they are not zero.

Taking the decoupling properties of g into account, it follows that in adapted coordinates: $\partial J^i_j/\partial x^a = \partial J^a_b/\partial y^i = 0$ and

$$J^a_{i|k} = \frac{\partial J^a_i}{\partial y^k} - J^a_j \Gamma^j_{ik} = \tfrac{1}{2}\alpha_b g^{ba} g_{ik}, \qquad J^i_{b|a} = \frac{\partial J^i_b}{\partial x^a} - J^i_c \Gamma^c_{ba} = \tfrac{1}{2}\alpha_j g^{ji} g_{ab}. \tag{4.17}$$

Hence, the different blocks of J have the following type of restricted dependence on the adapted coordinates:

$$J_1 = J^i_j(y)\frac{\partial}{\partial y^i} \otimes dy^j, \quad J_2 = J^a_b(x)\frac{\partial}{\partial x^a} \otimes dx^b,$$

while

$$J_{21} = J^a_i(y,x)\frac{\partial}{\partial x^a} \otimes dy^i, \quad J_{12} = J^i_a(y,x)\frac{\partial}{\partial y^i} \otimes dx^a,$$

with

$$\frac{\partial J^a_i}{\partial y^k} = \frac{\partial J^a_k}{\partial y^i} \quad \text{and} \quad \frac{\partial J^i_a}{\partial x^b} = \frac{\partial J^i_b}{\partial x^a}. \tag{4.18}$$

Define $\mu_1 := P_1(\mu) = Q_i(y)dy^i$ and $\mu_2 := P_2(\mu) = Q_a(y,x)dx^a$.

Proposition 4.12. *The driving system of a driven cofactor system (g, μ, J) has itself a cofactor representation determined by (g_1, μ_1, J_1), assuming that J_1 is nonsingular.*

Proof. J_1 is a scKt with respect to g_1 since

$$J_{ij|k} = \tfrac{1}{2}(\alpha_{1i} g_{jk} + \alpha_{1j} g_{ik}), \qquad \alpha_1 = d(\operatorname{tr} J_1).$$

From the definition of D_J (4.3) and since (g, μ, J) is a cofactor system, we have

$$0 = D_J\mu = d_J\mu + d(\operatorname{tr} J) \wedge \mu = i_J d\mu - d(J\mu) + d(\operatorname{tr} J) \wedge \mu.$$

Applying this in particular to vector fields belonging to $K^\perp$ and taking the decomposition of J and μ into account, this reduces to: $\forall X, Y \in K^\perp$

$$\begin{aligned} 0 &= d\mu_1(J_1X, Y) + d\mu_1(X, J_1Y) - d(J_1\mu_1)(X,Y) + (d(\operatorname{tr} J_1) \wedge \mu_1)(X,Y) \\ &\quad + d\mu_2(J_{21}X, Y) + d\mu_2(X, J_{21}Y) - d(J_{21}\mu)(X,Y). \end{aligned}$$

The second line equals in adapted coordinates

$$\left(\frac{\partial J^a_i}{\partial y^k} - \frac{\partial J^a_k}{\partial y^i}\right) Q_a X^i Y^k$$

which is zero due to (4.18). The remaining terms then express that $D_{J_1}\mu_1 = 0$, thus the driving system is also a cofactor system. □

This implies in particular that this system has a quadratic first integral of cofactor type (see Theorem 4.4)

$$E^1 = \tfrac{1}{2}A^1_{ij}(y)\dot{y}^i\dot{y}^j + W^1(y), \tag{4.19}$$

where $A^1 = \operatorname{cof} J_1$. The function W^1 is determined by the relation $A^1\mu_1 = -dW^1$, which is locally equivalent to the condition $D_{J_1}\mu_1 = 0$ in the definition of a cofactor system.

Lemma 4.13. *The projection operators P_1 and P_2 formally satisfy the requirements for a scKt with respect to g.*

Proof. For $X \in K^\perp$ and $Y \in K$, the ∇-invariance of K and $K^\perp$ implies

$$\begin{aligned}
\nabla X &= \nabla(P_1(X)) = (\nabla P_1)(X) + P_1(\nabla X) = (\nabla P_1)(X) + \nabla X,\\
0 &= \nabla(P_1(Y)) = (\nabla P_1)(Y) + P_1(\nabla Y) = (\nabla P_1)(Y).
\end{aligned}$$

It follows easily that $\nabla P_1 = 0$. Similarly, one finds that $\nabla P_2 = 0$. If one thinks of their representation in adapted coordinates, it is obvious that they have constant trace: so $d(\operatorname{tr} P_i) = 0$. Therefore, $\nabla P_i = 0$ expresses that they formally satisfy the defining relation (1.10) of scKts. □

Proposition 4.14. *Let (g, μ, J) be a driven cofactor system. Then the full nonconservative system Γ has a second, degenerate cofactor representation with P_2 in the role of scKt.*

Proof. From the preceding lemma it follows that P_2 is a scKt with respect to g, and thus $N_{P_2} = 0$ (Theorem 1.30). Then (1.3) implies that

$$d\mu(P_2X, P_2Y) = d_{P_2}(P_2\mu)(X, Y).$$

Therefore, the hypothesis $d\mu(K,K) = 0$ becomes equivalent with $d_{P_2}(P_2\mu) = 0$. In addition, thinking in the adapted coordinates (y^i, x^a), $P_1\mu$ is a 1-form involving the coordinates y^i of the driving system only, whence the same is true for its exterior derivative. It follows that $i_{P_2}d(P_1\mu) = 0$ and since $P_2(P_1\mu) = 0$ trivially, also $d_{P_2}(P_1\mu) = (i_{P_2}d - di_{P_2})(P_1\mu) = 0$. Hence, we can conclude that $d_{P_2}\mu = 0$, which is the same as $D_{P_2}\mu = 0$ in view of P_2 having constant trace. □

It follows that we have a kind of degenerate cofactor pair system. The implications of which we will investigate now. The starting point is that $J + aP_2$ also satisfies the scKt condition for any constant a (and is nonsingular for sufficiently small values of a). Let $A(a)$ denote the cofactor tensor of $J + aP_2$, depending on the parameter $a \in \mathbb{R}$, so that

$$(J + aP_2)A(a) = A(a)(J + aP_2) = \det(J + aP_2)I_N. \tag{4.20}$$

Since $P_2 = I_n$ in adapted coordinates, it is clear that $A(a)$ and $\det(J + aP_2)$ are polynomials in a of degree n. We represent them as follows,

$$A(a) = \sum_{i=1}^{n+1} A_{(i)}a^{i-1}, \qquad \det(J + aP_2) = \sum_{i=1}^{n+1} \Delta_{(i)}a^{i-1},$$

and identify the coefficients of equal powers of a in the identity (4.20). We get

$$P_2A_{(n+1)} = A_{(n+1)}P_2 \;=\; 0, \tag{4.21}$$

$$JA_{(i+1)} + P_2A_{(i)} = A_{(i+1)}J + A_{(i)}P_2 \;=\; \Delta_{(i+1)}I_N, \qquad (1 \leq i \leq n), \tag{4.22}$$

$$JA_{(1)} = A_{(1)}J \;=\; \Delta_{(1)}I_N. \tag{4.23}$$

Note first that the relation (4.23) expresses that $A_{(1)} = \operatorname{cof} J$ and $\Delta_{(1)} = \det J$. Information about the block structure of the different $A_{(i)}$ should follow by left and right actions of the projectors P_k on these relations. It immediately follows from (4.21), with similar notations as is (4.15), for example that

$$A_{(n+1)21} = A_{(n+1)12} = A_{(n+1)2} = 0. \tag{4.24}$$

Acting with P_1 on both sides of the relation (4.22) with $i = n$ subsequently implies that

$$J_1A_{(n+1)1} = A_{(n+1)1}J_1 = \Delta_{(n+1)}P_1,$$

or in the restriction to $K^{\perp}$:

$$J_1A_{(n+1)1} = A_{(n+1)1}J_1 = \Delta_{(n+1)}I_m.$$

Taking into account that terms of degree n in $A(a)$ can only be produced by minors of the J_1 elements of J, this implies that

$$A_{(n+1)1} = A^1 = \operatorname{cof} J_1 \qquad \text{and} \qquad \Delta_{(n+1)} = \det J_1. \tag{4.25}$$

The same equation (4.22) for $i = n$ then further yields information about parts of $A_{(n)}$. By acting with P_2 on both sides of (4.22) we find $A_{(n)2} = (\det J_1)I_n$. A left action of P_1 and a right action of P_2 implies $(\operatorname{cof} J_1)J_{12} + A_{(n)12} = 0$. Likewise, a left action of P_2 and a right action of P_1 fixes $A_{(n)21}$: $J_{21}(\operatorname{cof} J_1) + A_{(n)21} = 0$. So three of the four parts of $A_{(n)}$ are determined

$$A_{(n)2} = (\det J_1)I_n, \qquad A_{(n)21} = -J_{21}(\operatorname{cof} J_1), \qquad A_{(n)12} = -(\operatorname{cof} J_1)J_{12}. \quad (4.26)$$

For the remaining part of $A_{(n)}$, we have to move to the next line in the hierarchy ($i = n-1$ in (4.22)): a right and left action of P_1 leads to $J_1 A_{(n)1} + J_{12}A_{(n)21} = \Delta_{(n)}P_1$, from which it readily follows that

$$A_{(n)1} = (\det J_1)J_1^{-1}J_{12}J_{21}J_1^{-1} + \Delta_{(n)}J_1^{-1}. \quad (4.27)$$

We will come back in more detail to the continuation of this recursive scheme when we are in a position to gather information about the functions $\Delta_{(i)}$ (see Section 4.7).

The important feature about having a cofactor pair scheme is that it gives rise to a family of quadratic first integrals which are in involution with respect to a double Poisson structure (see Theorem 4.6). We sketch how this will work here by following the procedure explained in [19]. According to Theorem 4.4, having a double cofactor representation with scKts such as J and P_2 in our present situation, will entail that $\hat{\Gamma}$ (4.5) satisfies a relation of the form

$$(\det(J + aP_2))\hat{\Gamma} = -P_{J+aP_2}dH(a), \quad \text{with} \quad H(a) = \tfrac{1}{2}A^{\alpha\beta}(a)p_\alpha p_\beta + W(a), \quad (4.28)$$

where $H(a)$ is a function on T^*M, depending on the parameter a. Since $P_{J+aP_2} = P_J + aP_{P_2}$, one can see that $H(a)$ will be a polynomial of degree at most n in the parameter a, say of the form

$$H(a) = \sum_{i=1}^{n+1} H_{(i)}a^{i-1}, \quad \text{with} \quad H_{(i)} = \tfrac{1}{2}A^{\alpha\beta}_{(i)}p_\alpha p_\beta + W_{(i)}, \quad (4.29)$$

where $A^{\alpha\beta}_{(i)}$ comes from the $A_{(i)}$ tensor considered before, with an index raised by the given metric g, and $W_{(i)}$ is a C^∞-function on M, determined by the relation $A_{(i)}\mu = -dW_{(i)}$. Naturally, $H(a)$ and therefore all its coefficients $H_{(i)}$, will be first integrals of the system $\hat{\Gamma}$.

Before proceeding, observe that $H_{(n+1)}$ is essentially the function E^1 (4.19) which is a first integral of the driving system. Indeed, it follows from the block structure of $A_{(n+1)}$ determined before that in adapted coordinates, with $p_i = g_{ij}(y)\,\dot{y}^j$,

$$H_{(n+1)} = \tfrac{1}{2}(A^1)^{ij}(y)p_ip_j + W^1(y). \tag{4.30}$$

Taking the various polynomial representations into account, property (4.28) becomes

$$(\textstyle\sum_{i=1}^{n+1}\Delta_{(i)}a^{i-1})\hat{\Gamma} = -(P_J + aP_{P_2})(\textstyle\sum_{i=1}^{n+1}dH_{(i)}a^{i-1}).$$

From the coordinate expression of P_J for a general J in (4.4), it follows that, in adapted coordinates,

$$P_{P_2} = \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial p_a}, \tag{4.31}$$

where the momentum variables p_a are defined by $p_a = g_{ab}(x)\dot{x}^b$. Identifying coefficients of equal powers of a requires first of all that we should have $P_{P_2}(dH_{(n+1)}) = 0$, and this is easily verified in view of the preceding observations. We further must have that

$$\Delta_{(i)}\hat{\Gamma} = -P_J(dH_{(i)}) - P_{P_2}(dH_{(i-1)}), \qquad 1 < i \leq n+1, \tag{4.32}$$

and finally for $i = 1$ that

$$\Delta_{(1)}\hat{\Gamma} = -P_J(dH_{(1)}),$$

but this is merely a confirmation of the quasi-Hamiltonian structure coming from J, since $\Delta_{(1)} = \det J$ and $H_{(1)} = H$ (see (4.6)).

There is no reason to expect that the n first integrals $H_{(1)}$ up to $H_{(n)}$ of $\hat{\Gamma}$ would depend on the coordinates (x^a, p_a) only. Nevertheless, they will be first integrals of the driven system along solutions $(y^i(t), p_i(t))$ of the driving system. As for the question of involutiveness, if we adopt the notational convention that $P_J(df) = \{f, \cdot\}_J$, it follows from (4.32) that

$$\{H_{(i)}, H_{(l)}\}_J + \{H_{(i-1)}, H_{(l)}\}_{P_2} = -\Delta_{(i)}\hat{\Gamma}(H_{(l)}) = 0, \quad 1 < i \leq n+1, 1 \leq l \leq n+1, \tag{4.33}$$

and from the two other observations about $H_{(n+1)}$ and $H_{(1)}$ that

$$\{H_{(1)}, H_{(l)}\}_J = \{H_{(n+1)}, H_{(l)}\}_{P_2} = 0, \qquad 1 \leq l \leq n+1. \tag{4.34}$$

Using (4.34), it further follows from (4.33) with $l = 1$ and $l = n+1$, respectively, that also

$$\{H_{(i)}, H_{(1)}\}_{P_2} = \{H_{(i)}, H_{(n+1)}\}_J = 0, \qquad 1 \leq i \leq n+1. \tag{4.35}$$

Now, for example, take (4.33) again with $i = n+1$: the first term is zero, so we conclude that $\{H_{(n)}, H_{(l)}\}_{P_2} = 0$ for all l. But then, taking $l = n$ in (4.33), the second term will be zero, so that we get $\{H_{(i)}, H_{(n)}\}_J = 0$ for $i = 2, \dots, n+1$ (and of course also for $i = 1$). The next step will be to express (4.33) for $i = n$ and subsequently for $l = n-1$ and so on. The final conclusion will be that all $H_{(i)}$ are in involution with respect to both the J-bracket and the P_2-bracket. It is worth observing (see (4.31)) that in adapted coordinates the P_2-bracket formally looks like the standard Poisson bracket in the (x^a, p_a) coordinates so that we have, along solutions of the driving system, n first integrals for the driven system which are in involution in the standard sense.

4.4 A special conformal Killing tensor for the metric of the driven system

Part of our basic assumptions so far is that both J and J_1 are nonsingular. When talking about nonsingularity here, don't forget the notational convention specified before. To say that J_1 is nonsingular of course only makes sense when we mean that $J_1|_{K^\perp}$ is nonsingular. We shall see now that these assumptions naturally lead to the introduction of another nonsingular type $(1,1)$ tensor field $\bar{J}_2$, which is a kind of deformation of J_2. This $\bar{J}_2$ turns out to play an important role in the establishment of the appropriate canonical transformations in order to reduce the Hamilton-Jacobi problem of the driven system into that of an equivalent autonomous system of Stäckel type.

For later reference, we look for a moment at the adjoint action of J on 1-forms. In doing so we use the same notation again. One has to keep in mind, however, that when compositions are involved (as in the definition of J_{12} and J_{21}), the order of such compositions has to be reversed. Let α be an arbitrary 1-form on M and put $\beta = J\alpha = J(P_1\alpha + P_2\alpha)$. To solve such a relation for α in terms of β, it is natural to actually compute $P_1\alpha$ and $P_2\alpha$. We first have for $P_1\beta$ and $P_2\beta$

$$P_1\beta = J_1(P_1\alpha) + J_{21}(P_2\alpha),$$
$$P_2\beta = J_{12}(P_1\alpha) + J_2(P_2\alpha).$$

Since J_1 is nonsingular, it follows from the first relation that

$$P_1\alpha = J_1^{-1}(P_1\beta) - J_1^{-1}J_{21}(P_2\alpha), \tag{4.36}$$

and substitution of this result in the second relation leads to

$$\bar{J}_2(P_2\alpha) = (P_2 - J_{12}J_1^{-1}P_1)\beta, \tag{4.37}$$

where $\bar{J}_2$ is defined (for its action on 1-forms) as

$$\bar{J}_2 = J_2 - J_{12}J_1^{-1}J_{21}. \tag{4.38}$$

Obviously $\bar{J}_2$ vanishes on $K^\perp$. $P_2\alpha$ now can be obtained from (4.37) and substitution in (4.36) subsequently gives us $P_1\alpha$. In fact, what we are looking at here is the following typical factorization of a matrix with a block structure, this time written as representing the action of a type $(1,1)$ tensor field on vector fields:

$$J = \begin{pmatrix} J_1 & J_{12} \\ J_{21} & J_2 \end{pmatrix} = \begin{pmatrix} J_1 & 0 \\ J_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & J_1^{-1}J_{12} \\ 0 & J_2 - J_{21}J_1^{-1}J_{12} \end{pmatrix}. \tag{4.39}$$

It follows that $\det J = (\det J_1)(\det \bar{J}_2)$, so $\bar{J}_2$ will be nonsingular on K and has components

$$\bar{J}_2{}^a_b = J^a_b - J^a_i(J_1^{-1})^i_j J^j_b. \tag{4.40}$$

Proposition 4.15. *$\bar{J}_2$ is a (parameter dependent) special conformal Killing tensor for g_2, and its cofactor tensor is $(\det J_1)^{-1}A_2$ with $A = \operatorname{cof} J$.*

Proof. The proof is a straightforward computation, for which it will be suitable to work in the adapted (y^i, x^a) coordinates. Let us lower an index to apply the scKt condition in the covariant form (1.10). Keeping in mind that $g_{ai} = g_{ia} = 0$, we have

$$\bar{J}_{2cb} := g_{ca}\bar{J}_2{}^a_b = J_{cb} - J_{ci}(J_1^{-1})^{ij}J_{jb}.$$

Since J is symmetric, the same is true for $\bar{J}_2$. Note that $\bar{J}_2$ depends on both sets of coordinates, but the y^i are regarded as external parameters for our present considerations. Since J_1 depends on the y^i only and the same of course holds for its inverse (or its cofactor tensor $A^1 = A_{(n+1)1}$), we get in the first place that

$$\bar{J}_{2cb|a} = J_{cb|a} - J_{ci|a}(J_1^{-1})^{ij} J_{jb} - J_{ci}(J_1^{-1})^{ij} J_{jb|a}.$$

It follows from (1.10) that

$$J_{cb|a} = \tfrac{1}{2}(\alpha_c g_{ba} + \alpha_b g_{ca}), \quad \text{and} \quad J_{ci|a} = \tfrac{1}{2}\alpha_i g_{ca}. \tag{4.41}$$

We then easily obtain that

$$\bar{J}_{2cb|a} = \tfrac{1}{2}(\bar{\alpha}_c g_{ba} + \bar{\alpha}_b g_{ca}), \quad \text{with} \quad \bar{\alpha}_c = \alpha_c - J_{ci}(J_1^{-1})^{ij}\alpha_j, \tag{4.42}$$

which shows that $\bar{J}_2$ is a scKt for g_2.

If we denote cof J as in (4.6) by A (which, as we have seen, is equal to $A_{(1)}$), we know that $(\det J) g^{ac} = J^a_\beta A^{\beta c} = J^a_i A^{ic} + J^a_b A^{bc}$ and $0 = (\det J) g^{jc} = J^j_k A^{kc} + J^j_b A^{bc}$. It then follows that

$$\begin{aligned}
\bar{J}^a_{2b} A^{bc} (\det J_1)^{-1} &= (\det J_1)^{-1}\left(J^a_b A^{bc} - J^a_i (J_1^{-1})^i_j J^j_b A^{bc}\right)\\
&= (\det J_1)^{-1}\left((\det J) g^{ac} - J^a_i A^{ic} + J^a_i (J_1^{-1})^i_j J^j_k A^{kc}\right)\\
&= (\det J_1)^{-1}(\det J) g^{ac}\\
&= (\det \bar{J}_2) g^{ac}
\end{aligned}$$

which proves the last statement of the proposition. □

Note that $\bar{J}_2$, perhaps rather unexpectedly, does not give rise to a cofactor system representation of the driven system in a strict sense. In other words, it is not true that the forces μ_2 of the driven system have the property $D_{\bar{J}_2}\mu_2 = 0$ or, equivalently, that (cof $\bar{J}_2)\mu_2$ is closed. At this moment, the closest we can get to such a property is that an associated driven cofactor system $(g_2, \bar{\mu}_2, \bar{J}_2)$ exists, with modified non-conservative forces $\bar{\mu}_2$.

Proposition 4.16. *A cofactor system, parametrically depending on the coordinates of the driving system, is determined by the triple $(g_2, \bar{\mu}_2, \bar{J}_2)$, where g_2 and $\bar{J}_2$ are as before, and*

$$\bar{\mu}_2 = P_2(dW_{(n)}), \tag{4.43}$$

$W_{(n)}$ being the function encountered in the recursive scheme following from (4.28).

Proof. We know that the given nonconservative forces μ satisfy the relation $A\mu = -dW$ or, equivalently, $(\det J)\mu = -J(dW)$. Consider the scheme at the beginning of this section which led to the introduction of $\bar{J}_2$. Letting $-dW$ play the role of α and $(\det J)\mu$ the role of β, we put $P_i(dW) = d_iW$ for convenience. The relation (4.37) becomes

$$-\bar{J}_2(d_2W) = (\det J)\mu_2 - (\det J)J_{12}J_1^{-1}\mu_1.$$

Taking into account that μ_1 satisfies $(\det J_1)\mu_1 = -J_1(d_1W^1)$ in view of the cofactor representation of the driving system, and that $\det J = (\det J_1)(\det \bar{J}_2)$, it follows that

$$-\bar{J}_2(d_2W) = (\det J)\mu_2 + (\det \bar{J}_2)J_{12}(d_1W^1). \tag{4.44}$$

Secondly, projecting the relation $A_{(n)}\mu = -dW_{(n)}$ under P_2, we get in the first place that

$$A_{(n)2}\mu_2 + A_{(n)12}\mu_1 = -d_2W_{(n)}.$$

Using the information gathered about $A_{(n)}$ in (4.26) and the cofactor system property of the driving system which was just recalled, this immediately leads to

$$(\det J_1)\mu_2 + J_{12}(d_1W^1) = -d_2W_{(n)}. \tag{4.45}$$

Substituting this result in (4.44), we obtain the relation

$$(\det \bar{J}_2)d_2W_{(n)} = \bar{J}_2(d_2W), \tag{4.46}$$

which implies, with $\bar{\mu}_2 = d_2W_{(n)}$, that

$$(\operatorname{cof} \bar{J}_2)\bar{\mu}_2 = d_2W. \tag{4.47}$$

Together with the knowledge that $\bar{J}_2$ is a scKt with respect to g_2, this expresses that the triple $(g_2, \bar{\mu}_2, \bar{J}_2)$ determines a cofactor system. $\square$

It is worth emphasizing again that the cofactor representation of this modified driven system must also be seen as a statement about the second-order differential equations for the x^a, which parametrically depend on the y-coordinates. This is clear, for example, from the fact that the projected exterior derivative in the picture is d_2.

As announced in the beginning of this chapter, one of our main objectives is to explore the remarkable situation that the driven system, although being essentially time-dependent along solutions of the driving system, does give rise in the end to a Stäckel-type Hamilton-Jacobi separability anyway. This will require a supplementary assumption about existence of independent eigenfunctions of J. We want to show at this point that it is again the tensor $\bar{J}_2$ which is relevant for this purpose.

Consider the (degenerate kind of) eigenvalue equation $\det(J - \lambda P_2) = 0$, which is a polynomial equation of degree n for λ. Our basic assumption now is that this equation has n functionally independent solutions u^a, in the sense that the 1-forms $P_2(du^a)$ are linearly independent. If we think of the u^a as expressed in terms of the adapted coordinates (y^i, x^b), this amounts to saying that the Jacobian $(\partial u^a/\partial x^b)$ is nonsingular. From the identity $\det(J - u^a(y,x)P_2) \equiv 0$ for each fixed $u^a(y,x)$, it follows that

$$\begin{aligned} 0 &\equiv d\left(\det(J - u^a(y,x)P_2)\right) \\ &= d\left(\det(J - \lambda P_2)\right)|_{\lambda = u^a(y,x)} + \left.\frac{\partial(\det(J - \lambda P_2))}{\partial \lambda}\right|_{\lambda = u^a(y,x)} du^a. \end{aligned}$$

But since $J - \lambda P_2$ has vanishing Nijenhuis torsion, we know from (1.4) that

$$(J - \lambda P_2)\, d\left(\det(J - \lambda P_2)\right) = \det(J - \lambda P_2)\, d\,\mathrm{tr}(J - \lambda P_2)$$

for all λ. It then follows by acting with $(J - u^a(y,x)P_2)$ on the preceding identity that

$$\left.\frac{\partial(\det(J - \lambda P_2))}{\partial \lambda}\right|_{\lambda = u^a(y,x)} (J - u^a(y,x)P_2)\, du^a = 0,$$

and thus, since all eigenfunctions are assumed to be simple, that

$$(J - u^a(y,x)P_2)\, du^a = 0. \tag{4.48}$$

Proposition 4.17. *Assume that the equation* $\det(J - \lambda P_2) = 0$ *has n functionally independent eigenfunctions u^a. Then, du^a is an eigenform of J (in the sense of equation (4.48)) corresponding to the eigenvalue u^a. Moreover, the u^a are also eigenfunctions of $\bar{J}_2$, with $P_2(du^a)$ as corresponding eigenform.*

Proof. It remains to prove the statement about $\bar{J}_2$. For that, it suffices to go back once more to the analysis about $\beta = J\alpha$ at the beginning of this section, with du^a in the role of α and $u^a P_2(du^a)$ (no sum!) in the role of β. The relation (4.37) then says that

$$\bar{J}_2(P_2(du^a)) = u^a P_2(du^a),$$

which is precisely what we need. □

4.5 A symplectic view-point and Darboux coordinates

We start this section by identifying Darboux coordinates for the symplectic form associated to the special conformal Killing tensor of the complete system. They are obtained by suitably modifying the momenta. But we then gradually develop arguments to come to an even better selection of modified momenta, which takes the specific decoupling properties of our system into account and are shown to be related to a time-dependent (standard) canonical transformation for the driven part of the system.

Since J is assumed to be nonsingular, the Poisson tensor associated to P_J actually comes from a symplectic form which we call ω_J. The sign convention which we adopted is that for any function F and nonsingular Poisson tensor P_J

$$X = -P_J(dF) \quad \Longleftrightarrow \quad i_X\omega_J = -dF.$$

One easily verifies that, referring to the general coordinate expression (4.4) of P_J, ω_J is given by

$$\omega_J = {J^{-1}}^{\alpha}_{\beta} dp_\alpha \wedge dq^\beta - \tfrac{1}{2} p_\gamma \left(\frac{\partial J^\gamma_\alpha}{\partial q^\beta} - \frac{\partial J^\gamma_\beta}{\partial q^\alpha} \right) {J^{-1}}^{\beta}_{\sigma} {J^{-1}}^{\alpha}_{\rho} dq^\sigma \wedge dq^\rho. \tag{4.49}$$

So far, this correspondence is valid for any nonsingular type $(1,1)$ tensor field J on M. The first term in ω_J strongly suggests introducing new momentum variables $\check{p}_\alpha$ by

$$p_\alpha = J^\beta_\alpha(q)\check{p}_\beta.$$

Lemma 4.18. *The coordinate change $(q,p) \leftrightarrow (q,\check{p})$ determines a Darboux chart for ω_J if and only if $N_J = 0$.*

Proof. From $\check{p}_\beta = {J^{-1}}^{\alpha}_{\beta} p_\alpha$, it is easy to compute that

$$d\check{p}_\beta \wedge dq^\beta = {J^{-1}}^{\alpha}_{\beta} dp_\alpha \wedge dq^\beta - \tfrac{1}{2}\check{p}_\delta \left(\frac{\partial J^\delta_\tau}{\partial q^\sigma} {J^{-1}}^{\tau}_{\rho} - \frac{\partial J^\delta_\tau}{\partial q^\rho} {J^{-1}}^{\tau}_{\sigma} \right) dq^\sigma \wedge dq^\rho.$$

Subtracting this from (4.49), one obtains the resulting 2-form

$$\tfrac{1}{2}\check{p}_\delta \left(J^\delta_\gamma \left(\frac{\partial J^\gamma_\alpha}{\partial q^\beta} - \frac{\partial J^\gamma_\beta}{\partial q^\alpha} \right) {J^{-1}}^{\beta}_{\sigma} {J^{-1}}^{\alpha}_{\rho} + \left(\frac{\partial J^\delta_\tau}{\partial q^\sigma} {J^{-1}}^{\tau}_{\rho} - \frac{\partial J^\delta_\tau}{\partial q^\rho} {J^{-1}}^{\tau}_{\sigma} \right) \right) dq^\sigma \wedge dq^\rho$$

which will be zero if all the coefficients vanish. Multiplying the coefficients with $J^\sigma_\mu J^\rho_\lambda$ gives

$$\tfrac{1}{2}\check{p}_\delta \left(J^\delta_\gamma \left(\frac{\partial J^\gamma_\lambda}{\partial q^\mu} - \frac{\partial J^\gamma_\mu}{\partial q^\lambda} \right) + \frac{\partial J^\delta_\lambda}{\partial q^\sigma} J^\sigma_\mu - \frac{\partial J^\delta_\mu}{\partial q^\rho} J^\rho_\lambda \right).$$

As we recognize herein the coefficient of N_J with respect to the standard coordinate basis (1.5), this is zero if and only if $N_J = 0$. □

As explained in Section 4.1, the fact that the SODE Γ on TM satisfies the requirements of a cofactor system is equivalent to saying that its image $\hat\Gamma$ under the Legendre map has a quasi-Hamiltonian representation (4.6) with respect to P_J. This in turn translates within the symplectic view-point to $(\det J) i_{\hat\Gamma}\omega_J = -dH$. The preceding lemma then says that ω_J will take the form of the standard symplectic form on T^*M when expressed in the variables $(q, \check{p})$. However, it is not clear that we can take advantage of such a coordinate change because it does not take account of the special feature of partial decoupling which our system exhibits. We shall show that there is a better choice of new momenta, which is inspired by the above transition to Darboux coordinates but does take the extra features of a driven cofactor system into account.

Going back to the SODE Γ on TM, we can first pass to the coordinates (y^i, x^a) adapted to the complementary distributions $K^\perp$ and K, before passing to the quasi-Hamiltonian representation of $\hat\Gamma$. The projectors P_1 and P_2 have corresponding actions on TM through their complete lifts; they give rise to a partial splitting of Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$ say, as exhibited in the equations (4.11), (4.12). Likewise, the complete lifts $\tilde{P}_1$, $\tilde{P}_2$ of the projectors to T^*M produce a partial decoupling $\hat\Gamma = \hat\Gamma_1 + \hat\Gamma_2$, which in adapted coordinates is simply the effect of transforming (4.11), (4.12) to equivalent first-order equations by passing to the momenta $p_i = g_{ij}(y)\dot{y}^j$ and $p_a = g_{ab}(x)\dot{x}^b$. It is worth illustrating this in more detail as follows. Applying the overall Legendre map to the SODE Γ (4.13), we obtain

$$\hat\Gamma = g^{\alpha\beta} p_\beta \frac{\partial}{\partial q^\alpha} + \Gamma^\gamma_{\mu\alpha} g^{\mu\delta} p_\gamma p_\delta \frac{\partial}{\partial p_\alpha} + Q_\alpha \frac{\partial}{\partial p_\alpha}.$$

In adapted (y, x)-coordinates, in view of the way the components of g and the connection coefficients decouple (see (4.14) and its consequences), this expression becomes

$$\hat{\Gamma} = g^{ij}(y)p_j\frac{\partial}{\partial y^i} + \Gamma^k_{ij}(y)g^{il}(y)p_kp_l\frac{\partial}{\partial p_j} + Q_j(y)\frac{\partial}{\partial p_j}$$
$$+ g^{ab}(x)p_b\frac{\partial}{\partial x^a} + \Gamma^c_{ab}(x)g^{ad}(x)p_cp_d\frac{\partial}{\partial p_b} + Q_b(x,y)\frac{\partial}{\partial p_b}. \qquad (4.50)$$

The first line reflects the fact that the driving system has its own cofactor representation, i.e. satisfies

$$(\det J_1)\hat{\Gamma}_1 = -P_{J_1}(dH_{(n+1)}), \qquad (4.51)$$

with $H_{(n+1)}$ as in (4.30). Concerning the second line, we should take into account the extra assumption that the driven system has a standard Hamiltonian representation: as indicated before, the condition $d\mu(K,K)=0$ in Definition 4.11 expresses that the force terms Q_a of the driven system are derivable from a potential energy function $V(x,y)$ say, depending parametrically on the driving coordinates y^i. It is then clear that the second line simply expresses that

$$\hat{\Gamma}_2 = -P_{P_2}(dh), \quad \text{with} \quad h = \tfrac{1}{2}g^{ab}(x)p_ap_b + V(x,y), \qquad (4.52)$$

keeping in mind that P_{P_2} in adapted coordinates merely is the standard Poisson structure in the variables (x^a, p_a). With this splitting of $\hat{\Gamma}$ in mind, it looks more appropriate not to spoil the decoupled feature of the driving system by introducing Darboux coordinates for the overall symplectic structure ω_J. Instead, we can put $p_i = J_1{}^j_i\tilde{p}_j$, which will have the effect of introducing Darboux coordinates for ω_{J_1}. As for the driven part $\hat{\Gamma}_2$, let us first investigate in detail what the introduction of the momenta $\check{p}$ would do.

Formally we can regard the transformation formulas $p_\alpha = J^\beta_\alpha(q)\check{p}_\beta$ as representing a relation between 1-forms on M, of the type $\beta = J\alpha$ discussed at the beginning of Section 4.4. It then follows from the considerations leading to (4.37) that

$$\bar{J}_2{}^b_a\check{p}_b = p_a - J^i_a J_1^{-1}{}^j_i p_j. \qquad (4.53)$$

This suggests that the more relevant momentum variables for the driven system actually are $\tilde{p}_a := \bar{J}_2{}^b_a\check{p}_b$. The conclusion from this preliminary analysis is that we shall consider the following linear change of momenta

$$p_i = J_1{}^j_i\tilde{p}_j, \qquad (4.54)$$
$$p_a = \tilde{p}_a + J^i_a\tilde{p}_i. \qquad (4.55)$$

It turns out that the transition from p_a to $\tilde{p}_a$, viewed as time-dependent transformation along solutions of the driving system, actually represents a time-dependent canonical transformation for the driven system in the standard sense and hence is ideally suited to preserve the special assumption on that system. Indeed, in view of the second of the properties (4.18), we know that the components J_a^i of J can be written as $J_a^i = \partial\psi^i/\partial x^a$ for some functions $\psi^i(x,y)$. Defining $F(x,\tilde{p},t)$ by

$$F(x,\tilde{p},t) = x^a\tilde{p}_a + \Psi(x,t), \quad \text{with} \quad \Psi(x,t) = \psi^i(x,y(t))\tilde{p}_i(t), \tag{4.56}$$

we create a generating function of mixed type (depending on the old position variables x and new momenta $\tilde{p}$) for a standard canonical transformation $(x^a, p_a) \leftrightarrow (x^a, \tilde{p}_a)$, which does not change the coordinates, transforms the momenta according to (4.55), but must be viewed as time-dependent along solutions of the driving system. The Hamiltonian of the transformed system then is given by

$$\tilde{h}(x,\tilde{p},t) := h + \frac{\partial F}{\partial t}. \tag{4.57}$$

We shall see in the next section that this canonical transformation is one of two steps which are required to relate the original Hamiltonian h of the driven system to the first integral $H_{(n)}$ of $\hat{\Gamma}$ and that $H_{(n)}$ is the key to understand the subtle way in which the driven system in the end corresponds to an autonomous Hamiltonian system which is separable in the Hamilton-Jacobi sense.

4.6 Separability of the Hamilton-Jacobi equation for the driven system

Let us start by computing the function $H_{(n)}$ expressed in the variables $(y^i, x^a, \tilde{p}_i, \tilde{p}_a)$. In view of (4.29), we have

$$H_{(n)} = \tfrac{1}{2}A^{ab}_{(n)}p_ap_b + A^{ai}_{(n)}p_ap_i + \tfrac{1}{2}A^{ij}_{(n)}p_ip_j + W_{(n)}.$$

From (4.26), raising an index, we learn that

$$A^{ab}_{(n)} = (\det J_1)g^{ab}, \qquad A^{ai}_{(n)} = -A^{1}{}^{i}_{l}J^{la}.$$

Making the substitutions (4.54), (4.55) it then readily follows (remember that A^1 is cof J_1) that

$$H_{(n)} = \tfrac{1}{2}(\det J_1)\, g^{ab}\tilde{p}_a\tilde{p}_b - \tfrac{1}{2}(\det J_1)J^{bi}J^j_b\tilde{p}_i\tilde{p}_j + \tfrac{1}{2}A^{kl}_{(n)}J_1{}^i_kJ_1{}^j_l\tilde{p}_i\tilde{p}_j + W_{(n)}.$$

Using (4.27) now we arrive at the following result:

$$H_{(n)} = \tfrac{1}{2}(\det J_1)\, g^{ab}\tilde{p}_a\tilde{p}_b + \tfrac{1}{2}\Delta_{(n)} J^{ij}\tilde{p}_i\tilde{p}_j + W_{(n)}. \tag{4.58}$$

So, introducing the new variables has the interesting effect of eliminating the terms in mixed momenta in $H_{(n)}$. Obviously, however, the effect on h will be the opposite. We get

$$h = \tfrac{1}{2}g^{ab}\tilde{p}_a\tilde{p}_b + J^{ai}\tilde{p}_a\tilde{p}_i + \tfrac{1}{2}J^{ai}J^j_a\tilde{p}_i\tilde{p}_j + V. \tag{4.59}$$

But for the interpretation as time-dependent canonical transformation, we need to look at the function $\tilde{h}$ (4.57) and will show now that this function is more closely related to $H_{(n)}$.

Lemma 4.19. *Under the canonical transformation with generating function (4.56), the transformed Hamiltonian $\tilde{h}$ of the driven system takes the following form, to within an additive function of time,*

$$\tilde{h} = (\det J_1)^{-1}H_{(n)} + J^{ai}\tilde{p}_a\tilde{p}_i. \tag{4.60}$$

Proof. We need to add to the expression (4.59) for h the term

$$\frac{\partial F}{\partial t} = \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial y^k}\,\dot{y}^k + \psi^k\dot{\tilde{p}}_k,$$

computed along solutions of the driving equations. Since the $\tilde{p}_i$ were introduced to provide Darboux coordinates, the Poisson tensor P_{J_1} in (4.51) takes the form of the standard Poisson tensor

$$P_{J_1} = \frac{\partial}{\partial y^k} \wedge \frac{\partial}{\partial \tilde{p}_k}.$$

So the equations of the driving system which satisfies (4.51), become

$$(\det J_1)\,\dot{y}^k = \frac{\partial H_{(n+1)}}{\partial \tilde{p}_k}, \qquad (\det J_1)\,\dot{\tilde{p}}_k = -\frac{\partial H_{(n+1)}}{\partial y^k},$$

whereby the function $H_{(n+1)}$ from (4.30), when expressed in the new momenta, reads

$$H_{(n+1)} = \tfrac{1}{2}(\det J_1)J_1^{ij}\tilde{p}_i\tilde{p}_j + W^1. \tag{4.61}$$

So we find

$$\begin{aligned}
\dot{y}^k &= J_1^{kj}\tilde{p}_j \\
\dot{\tilde{p}}_k &= -\tfrac{1}{2}\frac{\partial J_1^{ij}}{\partial y^k}\tilde{p}_i\tilde{p}_j - \tfrac{1}{2}(\det J_1)^{-1}\frac{\partial(\det J_1)}{\partial y^k}J_1^{ij}\tilde{p}_i\tilde{p}_j - (\det J_1)^{-1}\frac{\partial W^1}{\partial y^k}.
\end{aligned}$$

It is now fairly straightforward to compute that $\tilde{h}$ can be written as,

$$\begin{aligned}
\tilde{h} = h &+ \frac{\partial\psi^i}{\partial y^k}J_1^{kj}\tilde{p}_i\tilde{p}_j + J_1^{j\,l}\Gamma^i_{kl}\psi^k\tilde{p}_i\tilde{p}_j - \tfrac{1}{2}J^{ij}_{1\;|k}\psi^k\tilde{p}_i\tilde{p}_j \\
&- \tfrac{1}{2}(\det J_1)^{-1}\frac{\partial \det J_1}{\partial y^k}\psi^k J_1^{ij}\tilde{p}_i\tilde{p}_j - (\det J_1)^{-1}\psi^k\frac{\partial W^1}{\partial y^k},
\end{aligned}\tag{4.62}$$

with h given by (4.59). Concerning the function we want to match it with, we first of all need more information about $\Delta_{(n)}$. Using the representation (4.39) of J, we can put

$$J + aP_2 = \begin{pmatrix} J_1 & 0 \\ J_{21} & 1 \end{pmatrix}\begin{pmatrix} 1 & J_1^{-1}J_{12} \\ 0 & \bar{J}_2 + aI_n \end{pmatrix},\tag{4.63}$$

from which it follows that

$$\sum_{i=1}^{n+1}\Delta_{(i)}a^{i-1} = \det(J + aP_2) = (\det J_1)\det(\bar{J}_2 + aI_n),\tag{4.64}$$

and hence that $\Delta_{(n)} = (\det J_1)(\operatorname{tr}\bar{J}_2)$. It follows from (4.58) that

$$(\det J_1)^{-1}H_{(n)} = \tfrac{1}{2}\,g^{ab}\tilde{p}_a\tilde{p}_b + \tfrac{1}{2}(\operatorname{tr}\bar{J}_2)J^{ij}\tilde{p}_i\tilde{p}_j + (\det J_1)^{-1}W_{(n)}.\tag{4.65}$$

It looks by far not obvious that the expressions (4.62) and (4.65) would differ only by one term (up to irrelevant functions of time only). We shall compare them indirectly by computing their derivatives with respect to the x^a and $\tilde{p}_a$. The latter is easy and gives

$$\frac{\partial\tilde{h}}{\partial\tilde{p}_a} = g^{ab}\tilde{p}_b + J^{ai}\tilde{p}_i, \qquad \frac{\partial}{\partial\tilde{p}_a}\left((\det J_1)^{-1}H_{(n)}\right) = g^{ab}\tilde{p}_b,$$

which is in line with the result we want to prove. For the other derivatives, the computations can be written in a somewhat more compact form if we use the linearly independent vector fields

$$X_a = \frac{\partial}{\partial x^a} + \Gamma^c_{ab}\tilde{p}_c\frac{\partial}{\partial\tilde{p}_b}$$

adapted to the connection, rather than the coordinate derivatives with respect to x^a. One can verify, recalling that $J^k_a = \partial\psi^k/\partial x^a$, that $X_a(\tilde{h})$ can be written as

$$\begin{aligned}
X_a(\tilde{h}) &= \tfrac{1}{2} g^{dc}\Gamma^b_{ad}\tilde{p}_c\tilde{p}_b + \tfrac{1}{2} g^{bd}\Gamma^c_{ad}\tilde{p}_c\tilde{p}_b + J^{ci}\Gamma^b_{ac}\tilde{p}_b\tilde{p}_i + \tfrac{1}{2}\frac{\partial g^{cb}}{\partial x^a}\tilde{p}_c\tilde{p}_b + \frac{\partial J^b i}{\partial x^a}\tilde{p}_b\tilde{p}_i \\
&\quad + \tfrac{1}{2}\left(\frac{\partial J^{bi}}{\partial x^a}J^j_b + J^{bi}\frac{\partial J^j_b}{\partial x^a}\right)\tilde{p}_i\tilde{p}_j + \frac{\partial V}{\partial x^a} + \frac{\partial J^i_a}{\partial y^k}J_1^{kj}\tilde{p}_i\tilde{p}_j + J^k_a J_1^{jl}\Gamma^i_{kl}\tilde{p}_i\tilde{p}_j \\
&\quad - \tfrac{1}{2} J^k_a {J_1^{ij}}_{|k}\tilde{p}_i\tilde{p}_j - \tfrac{1}{2}(\det J_1)^{-1}\frac{\partial \det J_1}{\partial y^k}J^k_a J_1^{ij}\tilde{p}_i\tilde{p}_j - (\det J_1)^{-1}J^k_a\frac{\partial W^1}{\partial y^k} \\
&= {J^{bi}}_{|a}\tilde{p}_b\tilde{p}_i + \tfrac{1}{2}\left({J^{bi}}_{|a}J^j_b + J^{bi}J^j_{b|a}\right)\tilde{p}_i\tilde{p}_j + J^i_{a|k}J_1^{jk}\tilde{p}_i\tilde{p}_j - \tfrac{1}{2}J^k_a {J_1^{ij}}_{|k}\tilde{p}_i\tilde{p}_j \\
&\quad - \tfrac{1}{2}(\det J_1)^{-1}\frac{\partial \det J_1}{\partial y^k}J^k_a J_1^{ij}\tilde{p}_i\tilde{p}_j + \frac{\partial V}{\partial x^a} - (\det J_1)^{-1}J^k_a\frac{\partial W^1}{\partial y^k}.
\end{aligned}$$

Making use of the scKt properties of J and J_1, plus the property (1.4) for J_1, this expression considerably simplifies and finally reduces to

$$\begin{aligned}
X_a(\tilde{h}) &= \tfrac{1}{2}\alpha_k g^{ik}\tilde{p}_i\tilde{p}_a - \tfrac{1}{2}\alpha_l (J_1^{-1})^l_k J^k_a J_1^{ij}\tilde{p}_i\tilde{p}_j + \tfrac{1}{2}\alpha_a J_1^{ij}\tilde{p}_i\tilde{p}_j \\
&\quad + \frac{\partial V}{\partial x^a} - (\det J_1)^{-1}J^k_a\frac{\partial W^1}{\partial y^k},
\end{aligned}$$

where α as before stands for $d(\operatorname{tr} J)$. Note in passing that, for example, $J^{ij} \equiv J_1^{ij}$. The computation of $X_a\left((\det J_1)^{-1}H_{(n)}\right)$ is much easier and gives

$$X_a\left((\det J_1)^{-1}H_{(n)}\right) = \tfrac{1}{2}\bar{\alpha}_a J_1^{ij}\tilde{p}_i\tilde{p}_j + (\det J_1)^{-1}\frac{\partial W_{(n)}}{\partial x^a},$$

with $\bar{\alpha} = d(\operatorname{tr}\bar{J}_2)$. We need to make three more observations now. The first is that

$$X_a(J^{bi}\tilde{p}_i\tilde{p}_b) = {J^{bi}}_{|a}\tilde{p}_i\tilde{p}_b = \tfrac{1}{2}\alpha_k g^{ik}\tilde{p}_i\tilde{p}_a,$$

which takes account of the first term of $X_a(\tilde{h})$. Secondly, we recall the difference between $\bar{\alpha}_a$ and α_a, as obtained in (4.42), which makes that the first term of $X_a\left((\det J_1)^{-1}H_{(n)}\right)$ matches two terms of $X_a(\tilde{h})$. Finally, the terms not containing momenta also match as a result of (4.45), taking into account that $\mu_2 = -d_2 V$. The conclusion now is that

$$X_a(\tilde{h}) = X_a\left((\det J_1)^{-1}H_{(n)} + J^{bi}\tilde{p}_i\tilde{p}_b\right).$$

Remember that the two functions under consideration are regarded here as depending on the x^a, $\tilde{p}_a$ and time t (along solutions of the driving equations). Since their derivatives with respect to the X_a and $\frac{\partial}{\partial \tilde{p}_a}$ are the same and hence also their derivatives with respect to the x^a, the conclusion is that they are indeed equal up to an additive function of time. □

The idea now is to try to get rid of the second term on the right in (4.60) by a further, suitable canonical transformation. It is of some interest to look at this kind of question in all generality and to observe that it can be resolved indeed by a suitable point transformation, which necessarily must be time-dependent however. The notations used below in discussing this general question have nothing to do with any of the specific situations encountered so far.

Lemma 4.20. *Suppose that $H_1(q,p,t)$ and $H_2(q,p,t)$ are two functions which differ by terms linear in the p_i. Then, there exists a point transformation, $(q,p) \leftrightarrow (Q,P)$ say, such that the transformed Hamiltonian of the system with Hamiltonian H_1 becomes equal to the function H_2 expressed in the new variables.*

Proof. By assumption, we have $H_1 = H_2 + \rho^i(q,t)p_i$ for some functions ρ^i. If $F(q,P,t) = P_i Q^i(q,t)$ is the generating function of an as yet unspecified point transformation, we know that the transformed Hamiltonian of the system with Hamiltonian H_1 will be given by

$$\tilde{H}_1 = H_1 + \frac{\partial F}{\partial t} = H_2 + \rho^i \frac{\partial Q^j}{\partial q^i} P_j + \frac{\partial Q^j}{\partial t} P_j,$$

so that the desired effect requires that each of the Q^j satisfies

$$\frac{\partial Q^j}{\partial t} + \rho^i \frac{\partial Q^j}{\partial q^i} = 0.$$

In other words, we need n functionally independent first integrals of the equations $\dot{q}^i = \rho^i(q,t)$, which can be obtained in principle and is of course the same as saying that we have to integrate those equations. Note that even if the given ρ^i would not depend on time, this procedure can only work with a time-dependent canonical transformation. □

In the case of interest, we are looking at equation (4.60) where the linear terms are of the form

$$\rho^a(x,t)\tilde{p}_a = J^{ai}(x, y(t))\tilde{p}_i(t)\,\tilde{p}_a.$$

We can try to find first integrals of the equations $\dot{x}^a = \rho^a(x,t)$ of the form $u^a = u^a(y(t), x)$, i.e. which are such that the time-dependence originates from solutions

$y(t)$ of the driving equations. This means that we have $\dot{y}^k = J^{ki}\tilde{p}_i$, and the first integral condition becomes

$$\frac{\partial u^a}{\partial t} + \rho^b \frac{\partial u^a}{\partial x^b} = \tilde{p}_i \left(J^{ik} \frac{\partial u^a}{\partial y^k} + J^{ib} \frac{\partial u^a}{\partial x^b} \right) = 0. \tag{4.66}$$

We can now prove one of our main results, for which we go back to the supplementary assumption mentioned at the end of Section 4.4. The eigenfunctions $u^a(y,x)$ which were introduced there will now be used as new coordinates for the driven system, along solutions $y(t)$ of the driving system, and we will denote the corresponding conjugate momenta (for reference to the notations used in the Euclidean case in [41]) by s_a.

Proposition 4.21. *Assume that the equation* $\det(J - \lambda P_2) = 0$ *has n functionally independent solutions* $u^a(y,x)$*. Then, the canonical transformation* $(x^a, p_a) \leftrightarrow (x^a, \tilde{p}_a)$ *with generating function (4.56), followed by the canonical transformation* $(x^a, \tilde{p}_a) \leftrightarrow (u^a, s_a)$ *with generating function* $F(x,s,t) = s_a u^a(y(t), x)$*, has the effect of transforming the Hamiltonian* h *of the driven system into the function* $(\det J_1)^{-1} H_{(n)}$.

Proof. We know from Lemma 4.19 that the first step brings the Hamiltonian h into the form (4.60). According to Lemma 4.20, the second step will eliminate the linear terms in the momenta $\tilde{p}_a$ in (4.60), provided the functions $u^a(y,x)$ have the property of making the right-hand side of (4.66) vanish. But the u^a satisfy the relations (4.48), from which it follows by a left action of P_1 that

$$J_1 P_1(du^a) + J_{21} P_2(du^a) = 0 \quad \text{or} \quad \left(J_j^k \frac{\partial u^a}{\partial y^k} + J_j^b \frac{\partial u^a}{\partial x^b} \right) dy^j = 0. \tag{4.67}$$

The desired result now follows by raising an index. □

At this point, it is important to be aware of another general and in fact very simple result.

Lemma 4.22. *Assume that a Hamiltonian system has a Hamiltonian* $K(q,p,t)$ *of the form* $K = \gamma(t) H(q,p,t)$*, whereby* H *is a first integral of the system and* γ *is an arbitrary function of* t *only. Then,* H *in fact cannot explicitly depend on time and the time-dependent Hamilton-Jacobi equation for* $K(q,p,t)$ *reduces to the autonomous one for* H.

Proof. It is obvious that (with respect to the standard Poisson bracket) we have $\{K, H\} = \gamma\{H, H\} = 0$, so that the given property $\dot{H} = \{K, H\} + \partial H/\partial t = 0$ reduces to $\partial H/\partial t = 0$. The Hamilton-Jacobi equation for K then reads

$$\gamma(t)H\left(q, \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} = 0,$$

and looking for a complete solution $S(q, t, \alpha)$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$, of the form $S = W(q, \alpha) - \alpha_1 \int \gamma dt$ immediately reduces it to the Hamilton-Jacobi equation

$$H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_1$$

for the autonomous function H. $\square$

Since $\det J_1$, along solutions of the driving system, is a function of time only and $H_{(n)}$ is known to be a first integral of the driven system under the same circumstances, it is clear that the assumptions of Proposition 4.21 precisely bring us in a situation where we can draw the quite surprising conclusion that $H_{(n)}$ will no longer be time-dependent in the (u, s) coordinates and separability of the Hamilton-Jacobi equation for the driven system is essentially a matter of separability of $H_{(n)}$. It remains to convince ourselves that the Hamilton-Jacobi equation for a system with $H_{(n)}$ as Hamiltonian is indeed separable. Now comes a rather subtle point in the argumentation. The point is this: we want to test separability of $H_{(n)}$ by using criteria which have an intrinsic, i.e. coordinate independent meaning. As such, it is the function $H_{(n)}$ which matters, expressed in any kind of coordinates. Transformation formulas from one set of coordinates to another can then depend on external parameters, if need be, but should not be regarded as depending on time because time-dependent canonical transformations do more to the Hamiltonian function than just expressing it in the new variables.

To be concrete now, the sufficient conditions for separability which we want to invoke are intrinsic indeed: they comprise the existence of a special conformal Killing tensor J for the kinetic energy metric of the Hamiltonian, plus a corresponding condition for admissible potentials V. For example we consider the condition $d(JdV) = 0$ mentioned in [7] which, in this context, actually corresponds to the particular case of the cofactor condition $A\mu = -dW$ when the forces are conservative. It is well known (see e.g. [20]) that if the scKt involved in these conditions has functionally independent eigenfunctions u^a, then the latter are separation coordinates for the

Hamilton-Jacobi equation. It so happens that we needed these u^a-coordinates already in Proposition 4.21 to prove that the original Hamiltonian h of the driven system can be transformed into the function $H_{(n)}$ (up to a factor). But having shown in this way that Hamilton-Jacobi separability becomes a matter of the function $H_{(n)}$, it is more appropriate to put this function to the test in the set of canonical coordinates $(x^a, \tilde{p}_a)$ which naturally present themselves prior to introducing the separation coordinates. The delicate issue alluded to above is that, when we subsequently want to pass to the new variables (u^a, s_a) again, the interpretation for this part of the story is that the functions $u^a(y, x)$ are regarded then as depending parametrically on the y-coordinates of the driving system (not as a time-dependent transformation along solutions $y(t)$ of that system).

Going back to the expression (4.65) of $H_{(n)}$, taking into account that the function $H_{(n+1)}$ in (4.61) is actually a constant parameter (along solutions of the driving system) which we called E^1 in (4.19), we have that

$$H_{(n)} = \tfrac{1}{2}(\det J_1)\, g^{ab}\tilde{p}_a\tilde{p}_b + (\operatorname{tr}\bar{J}_2)(E^1 - W^1(y)) + W_{(n)}(y, x). \tag{4.68}$$

This is the right expression for activating our test because all the ingredients we need for that have been prepared in Section 4.4.

Proposition 4.23. *$\bar{J}_2$ is a special conformal Killing tensor for the metric associated to the quadratic terms in (4.68) and the remaining terms satisfy the conditions for an admissible potential for Hamilton-Jacobi separability.*

Proof. We know from Proposition 4.15 that $\bar{J}_2$ is a scKt for $g_2 = (g_{ab}(x))$ and recall that the characterizing property (1.10) of scKts, when expressed in terms of the underlying type $(1, 1)$ tensor field is given by (1.11), with $\alpha = d(\operatorname{tr} J)$. It is then clear that the same J is also a scKt with respect to any constant multiple of g. In the case of the quadratic terms in (4.68), we are precisely looking at a constant multiple of the metric g_2 since $(\det J_1)$ is a function of the external y-parameters only, whence the conclusion about $\bar{J}_2$. We further observed above that the condition for admissible potentials is a reduced form of the cofactor condition that $(\operatorname{cof} J)\mu$ should be closed. For the situation at hand we already know from Proposition 4.16 that the function $W_{(n)}$ satisfies this condition with respect to the scKt $\bar{J}_2$. It remains to show that the same is true for the remaining term in (4.68) which in fact, since the factor $(E^1 - W^1(y))$ can be treated as a constant here, amounts to saying that the function $\operatorname{tr}\bar{J}_2$ satisfies the condition. But this is trivially the case, because it follows from (1.4) that for any tensor J with vanishing Nijenhuis torsion, $(\operatorname{cof} J)d(\operatorname{tr} J) = d(\det J)$. □

We sum up the main results about the driven system now. We know from Section 4.3 that the driven system, along solutions of the driving system, has n first integrals $H_{(i)}$ which are in involution with respect to two Poisson structures, one of which is the standard one when using coordinates adapted to the integrable distributions K and $K^{\perp}$. Under the assumption that the characteristic equation $\det(J - \lambda P_2) = 0$ has n functionally independent solutions $u^a(y,x)$, we have seen in Proposition 4.21 that the given Hamiltonian h of the driven system, can be transformed into the function $(\det J_1)^{-1} H_{(n)}$. The fact that all the $H_{(i)}$ are first integrals then implies that

$$\frac{\partial H_{(i)}}{\partial t} + (\det J_1)^{-1}\{H_{(i)}, H_{(n)}\} = 0.$$

But, in view of the involutivity, this actually means that none of the first integrals will be time-dependent when expressed in the canonical coordinates (u^a, s_a). This is in line with the results of Proposition 4.23 which mainly says that due to the existence of a scKt $\bar{J}_2$, the Hamilton-Jacobi equation of the (autonomous) function $H_{(n)}$ is separable. In fact, referring to the results which can be found in [20] for example, we can state more precisely that we are looking at a separable system of Stäckel type and that the eigenfunctions $u^a(y,x)$ of $\bar{J}_2$ are orthogonal separation coordinates. It should then be true indeed that in those coordinates we have n time-independent quadratic first integrals in involution. We will illustrate this in the following section. Note that for the function $H_{(n)}$, for example, this time-independence means, among other things (see the expression (4.68)), that the function $W_{(n)} - (\operatorname{tr} \bar{J}_2)W^1$, when passing from the coordinates (y,x) to the coordinates (y,u) with $u = u(y,x)$, should become independent of the y-variables.

4.7 Explicit expressions for the tensors $A_{(i)}$ and the first integrals $H_{(i)}$

To give some further backing for all these rather subtle properties, we carry out some more explicit calculations. The idea is that we complete the recursive scheme started in Section 4.3. We will establish, in analogy with the expressions (4.26) and (4.27), explicit results for the different parts of all the $A_{(i)}$ and compute explicit expressions for the corresponding functions $H_{(i)}$.

Consider the recursive scheme for the tensors $A_{(i)}$, in particular the relations (4.22) for $i = 1, \ldots, n$. We have seen that taking $i = n$ led to determining relations for three of the four parts of $A_{(n)}$ (see (4.26)), while the remaining block $A_{(n)1}$ had to be obtained from taking subsequently $i = n - 1$. This procedure can be continued all the way down, and the essential features of the recursion are captured in the following statement.

Proposition 4.24. *Suppose that we know the tensor parts $A_{(i+1)21}$, $A_{(i+1)12}$ and $A_{(i+1)2}$ and that they satisfy the identities*

$$J_1 A_{(i+1)12} + J_{12} A_{(i+1)2} \equiv 0 \equiv A_{(i+1)21} J_1 + A_{(i+1)2} J_{21}, \tag{4.69}$$

$$J_{21} A_{(i+1)12} + J_2 A_{(i+1)2} \equiv A_{(i+1)21} J_{12} + A_{(i+1)2} J_2. \tag{4.70}$$

Then, the following are determining equations for the completion of the construction of $A_{(i+1)}$ and for the next step in the recursion

$$A_{(i+1)1} = \Delta_{(i+1)} J_1^{-1} - J_1^{-1} J_{12} A_{(i+1)21}, \tag{4.71}$$

$$A_{(i)21} = -J_{21} A_{(i+1)1} - J_2 A_{(i+1)21}, \tag{4.72}$$

$$A_{(i)12} = -A_{(i+1)1} J_{12} - A_{(i+1)12} J_2, \tag{4.73}$$

$$A_{(i)2} = \Delta_{(i+1)} P_2 - J_{21} A_{(i+1)12} - J_2 A_{(i+1)2}. \tag{4.74}$$

Moreover, the newly obtained parts of $A_{(i)}$ will satisfy the same three identities as those assumed for $A_{(i+1)}$, while the completion of $A_{(i+1)}$ will give rise to the following supplementary identity:

$$J_1 A_{(i+1)1} + J_{12} A_{(i+1)21} \equiv A_{(i+1)1} J_1 + A_{(i+1)12} J_{21}. \tag{4.75}$$

Proof. Consider the recursion relation (4.22) which in fact has two parts. The idea is to compose each of those parts on the left and on the right with one of the projectors P_i. This gives rise to a total of 8 equations. For example, acting with P_1 on both sides of the relation $JA_{(i+1)} + P_2 A_{(i)} = \Delta_{(i+1)} I_N$ (which we can call a (P_1, P_1) action for brevity) implies that we must have

$$J_1 A_{(i+1)1} + J_{12} A_{(i+1)21} = \Delta_{(i+1)} P_1,$$

from which the determining equation (4.71) follows. Likewise, a two-sided (P_2, P_1) action gives

$$J_{21} A_{(i+1)1} + J_2 A_{(i+1)21} + A_{(i)21} = 0,$$

which determines $A_{(i)21}$ by (4.72). The double P_2 action generates

$$J_{21}A_{(i+1)12} + J_2A_{(i+1)2} + A_{(i)2} = \Delta_{(i+1)}P_2,$$

from which (4.74) follows. The remaining (P_1, P_2) combination merely confirms the first of the assumed identities (4.69). Starting from the other part in (4.22), $A_{(i+1)}J + A_{(i)}P_2 = \Delta_{(i+1)}I_N$, the determining equation (4.73) follows from a (P_1, P_2) action,

$$A_{(i+1)1}J_{12} + A_{(i+1)12}J_2 + A_{(i)12} = 0.$$

The (P_2, P_1) combination confirms the second of the assumed identities in (4.69). The double P_2 action gives rise to another determining equation for $A_{(i)2}$:

$$A_{(i+1)21}J_{12} + A_{(i+1)2}J_2 + A_{(i)2} = \Delta_{(i+1)}P_2,$$

which is consistent with the first one in view of the identity (4.70). Finally the double P_1 action implies that,

$$A_{(i+1)1}J_1 + A_{(i+1)12}J_{21} = \Delta_{(i+1)}P_1,$$

and thus, again for consistency, $A_{(i+1)}$ must satisfy the identity (4.75). To conclude the full recursion step, we need to verify that the obtained blocks $A_{(i)21}$, $A_{(i)12}$ and $A_{(i)2}$ satisfy corresponding identities of the form (4.69) and (4.70) in view of those assumed for $A_{(i+1)}$. We will only show the first identity of (4.69), the other identities follow by a similar straightforward computation. We have

$$\begin{aligned}
J_1A_{(i)12} + J_{12}A_{(i)2} &= -J_1A_{(i+1)1}J_{12} - J_1A_{(i+1)12}J_2 + \Delta_{(i+1)}J_{12} \\
&\quad - J_{12}J_{21}A_{(i+1)12} - J_{12}J_2A_{(i+1)2} \\
&= J_{12}A_{(i+1)21}J_{12} - J_1A_{(i+1)12}J_2 - J_{12}J_{21}A_{(i+1)12} - J_{12}J_2A_{(i+1)2} \\
&= -J_{12}A_{(i+1)2}J_2 - J_1A_{(i+1)12}J_2 \\
&\equiv 0,
\end{aligned}$$

where we made use of (4.71), (4.73), (4.74) and as last (4.70). □

The identities which the different parts of all $A_{(i)}$ tensors satisfy are important to get to a considerable simplification of the recursive scheme. Indeed, as will be shown now, it turns out that knowledge of the block $A_{(i)2}$ suffices to determine the three other blocks of $A_{(i)}$. Moreover we can set up a recursive scheme to determine $A_{(i)2}$

from $A_{(i+1)2}$ and we shall see that this procedure brings the tensor $\bar{J}_2$ back into the spotlights.

It follows from the identities (4.69) that for each $i = 1, \dots, n$:

$$A_{(i)12} = -J_1^{-1} J_{12} A_{(i)2}, \tag{4.76}$$
$$A_{(i)21} = -A_{(i)2} J_{21} J_1^{-1}, \tag{4.77}$$

and subsequently from the determining relation (4.71) that

$$A_{(i)1} = \Delta_{(i)} J_1^{-1} + J_1^{-1} J_{12} A_{(i)2} J_{21} J_1^{-1}. \tag{4.78}$$

Making use of (4.76), (4.77) in the identity (4.70) it is easy to see that this expresses the commutativity

$$\bar{J}_2 A_{(i)2} = A_{(i)2} \bar{J}_2. \tag{4.79}$$

Finally, the determining equation (4.74) reduces to

$$A_{(i)2} = \Delta_{(i+1)} P_2 - \bar{J}_2 A_{(i+1)2}. \tag{4.80}$$

Lemma 4.25. *The blocks $A_{(i)2} = P_2 \circ A_{(i)} \circ P_2$ of the $A_{(i)}$ tensors which determine the quadratic part of the first integrals $H_{(i)}$ are recursively given by*

$$A_{(i)2} = \Delta_{(i+1)} P_2 + \sum_{j=1}^{n-i} (-1)^j \Delta_{(j+i+1)} \bar{J}_2{}^j, \quad i = 1, \dots, n-1, \tag{4.81}$$

and all other parts of $A_{(i)}$ follow from $A_{(i)2}$.

Proof. We know that $A_{(n)2} = \Delta_{(n+1)} P_2$, with $\Delta_{(n+1)} = \det J_1$. The recursive relation (4.81) then easily follows from (4.80) by induction. The last part of the statement has already been proved above. □

Obviously, we are now in a position to venture computing a more explicit expression for the functions $H_{(i)}$ and it turns out that it is most appropriate to do this in terms of the momenta $\tilde{p}$ again.

Proposition 4.26. *The quadratic first integrals $H_{(i)}$ (for $i = 1, \dots, n$) are given by*

$$H_{(i)} = \tfrac{1}{2} A_{(i)}^{ab} \tilde{p}_a \tilde{p}_b + \tfrac{1}{2} \Delta_{(i)} J_1^{kl} \tilde{p}_k \tilde{p}_l + W_{(i)}, \tag{4.82}$$

where

$$A_{(i)}^{ab} = \Delta_{(i+1)} g^{ab} + \sum_{j=1}^{n-i} (-1)^j \Delta_{(j+i+1)} (\bar{J}_2{}^j)^{ab}. \tag{4.83}$$

Proof. Recall that $H_{(n)}$ has already been computed (see (4.58)) and is indeed of the form (4.82). From (4.29), we further have that

$$H_{(i)} = \tfrac{1}{2}A^{ab}_{(i)}p_a p_b + A^{ak}_{(i)}p_a p_k + \tfrac{1}{2}A^{kl}_{(i)}p_k p_l + W_{(i)}.$$

Observe first that raising indices in (4.80) gives rise to the formula

$$A^{ab}_{(i)} = \Delta_{(i+1)}g^{ab} - \bar{J}_2{}^b_d A^{da}_{(i+1)},$$

and it subsequently follows from (4.76) and (4.78) that

$$\begin{aligned}
A^{ak}_{(i)} &= J_1^{-1}{}^k{}_l J^l_b(\bar{J}_2{}^b_d A^{da}_{(i+1)} - \Delta_{(i+1)}g^{ab})\\
A^{kl}_{(i)} &= \Delta_{(i)}J_1^{-1kl} - J_1^{-1}{}^l{}_j J^j_b \bar{J}_2{}^b_d A_{(i+1)}{}^d_c J^c_m J_1^{-1mk} + \Delta_{(i+1)}J_1^{-1}{}^l{}_j J^j_c J^c_m J_1^{-1mk}.
\end{aligned}$$

We now compute each of the quadratic parts of $H_{(i)}$ in terms of the $\tilde{p}$, using (4.54), (4.55). The first term becomes

$$\begin{aligned}
\tfrac{1}{2}A^{ab}_{(i)}p_a p_b = \tfrac{1}{2}A^{ab}_{(i)}\tilde{p}_a\tilde{p}_b &+ \Delta_{(i+1)}J^{bj}\tilde{p}_b\tilde{p}_j - \bar{J}_2{}^b_d A^{da}_{(i+1)}J^j_b\tilde{p}_a\tilde{p}_j\\
&+ \tfrac{1}{2}\Delta_{(i+1)}J^{bj}J^k_b\tilde{p}_j\tilde{p}_k - \tfrac{1}{2}\bar{J}_2{}^b_d A^{da}_{(i+1)}J^j_a J^k_b\tilde{p}_j\tilde{p}_k
\end{aligned}$$

where we made use of (4.79). For the second term we have

$$A^{ak}_{(i)}p_a p_k = \bar{J}_2{}^b_d A^{da}_{(i+1)}J^j_b\tilde{p}_a\tilde{p}_j - \Delta_{(i+1)}J^{bj}\tilde{p}_b\tilde{p}_j + \bar{J}_2{}^b_d A^{da}_{(i+1)}J^j_a J^k_b\tilde{p}_j\tilde{p}_k - \Delta_{(i+1)}J^{bj}J^k_b\tilde{p}_j\tilde{p}_k.$$

The last quadratic term of $H_{(i)}$ equals

$$\tfrac{1}{2}A^{kl}_{(i)}p_k p_l = \tfrac{1}{2}\Delta_{(i)}J_1^{kl}\tilde{p}_k\tilde{p}_l + \tfrac{1}{2}\Delta_{(i+1)}J^{bj}J^k_b\tilde{p}_j\tilde{p}_k - \tfrac{1}{2}\bar{J}_2{}^b_d A^{da}_{(i+1)}J^j_a J^k_b\tilde{p}_j\tilde{p}_k.$$

The formula (4.82) then readily follows, while (4.83) merely is the contravariant form of (4.81), and will turn out to be useful further on. □

As was the case with $H_{(n)}$, we recognize in the expression for the other $H_{(i)}$ part of the constant E^1 of the driving system. Explicitly, with the help of (4.19), it is clear that along solutions of the driving system, the $H_{(i)}$ can be written as

$$H_{(i)} = \tfrac{1}{2}A^{ab}_{(i)}\tilde{p}_a\tilde{p}_b + \frac{\Delta_{(i)}}{\det J_1}(E^1 - W^1) + W_{(i)}, \tag{4.84}$$

whereby we recall that $\det J_1 = \Delta_{(n+1)}$. We shall now finally illustrate that applying the parameter-dependent coordinate change $(x^a, \tilde{p}_a) \leftrightarrow (u^a, s_a)$ where $u^a = u^a(y, x)$ turns the expressions for the $H_{(i)}$ into functions which no longer depend on the parameters y^i and, hence, in their interpretation of first integrals of the driven system along solutions of the driving system become effectively time-independent quadratic first integrals.

According to Proposition 4.17, the u^a are eigenfunctions of $\bar{J}_2$, so that the coordinate expression of $\bar{J}_2$, in the variables (y, u) takes the simple form

$$\bar{J}_2 = u^a \frac{\partial}{\partial u^a} \otimes du^a.$$

Going back to the result (4.64), it follows that

$$\sum_{i=1}^{n+1} \Delta_{(i)} a^{i-1} = (\det J_1) \det(\bar{J}_2 + aI_n) = (\det J_1) \prod_{b=1}^{n} (u^b + a),$$

which in turn implies that

$$\Delta_{(i+1)} = (\det J_1)\, \sigma_{n-i}(u), \qquad i = 0, \ldots, n, \tag{4.85}$$

where $\sigma_j(u)$ denotes the elementary symmetric polynomial of order j defined by (1.16).

For information about the functional dependence of the other terms in the potential part of the $H_{(i)}$ we go back to the double cofactor representation of the overall system, which was the start of the recursive scheme in Section 4.3. We know that with $A(a)$ representing the cofactor tensor of $J + aP_2$, the force terms μ of the overall system satisfy the relation $A(a)\mu = -dW(a)$ or equivalently,

$$\det(J + aP_2)\mu = -(J + aP_2)dW(a).$$

Projecting this relation under P_1 it follows that

$$\det(J + aP_2)\mu_1 = -J_1 P_1(dW(a)) - J_{21} P_2(dW(a)). \tag{4.86}$$

We first derive the coordinate expression under the transformation $(y, x) \leftrightarrow (y, u)$ where $u = u(y, x)$ for the tensors involved. We obtain

$$P_1 = \left(\frac{\partial}{\partial y^i} + \frac{\partial u^b}{\partial y^i} \frac{\partial}{\partial u^b} \right) \otimes dy^i,$$

and for P_2, where we make use of the equality following from deriving the identity $u^a \equiv u^a(y, x(y, u))$ to u^c and y^j, we find

$$P_2 = \frac{\partial u^a}{\partial x^b} \frac{\partial}{\partial u^a} \otimes \left(\frac{\partial x^b}{\partial y^j} dy^j + \frac{\partial x^b}{\partial u^c} du^c \right) = \frac{\partial}{\partial u^a} \otimes du^a - \frac{\partial u^a}{\partial y^j} \frac{\partial}{\partial u^a} \otimes dy^j.$$

Similarly, J_1 and J_{21} are given by

$$J_1 = J^i_j \left(\frac{\partial}{\partial y^i} + \frac{\partial u^b}{\partial y^i} \frac{\partial}{\partial u^b} \right) \otimes dy^j, \qquad J_{21} = J^a_i \frac{\partial u^c}{\partial x^a} \frac{\partial}{\partial u^c} \otimes dy^i.$$

Then (4.86) acquires the coordinate expression

$$\det(J_1)\det(\bar{J}_2 + aP_2)\mu_1 = -\left(J^i_j \frac{\partial W(a)}{\partial y^i} + J^i_j \frac{\partial u^b}{\partial y^i} \frac{\partial W(a)}{\partial u^b} + J^c_j \frac{\partial u^b}{\partial x^c} \frac{\partial W(a)}{\partial u^b} \right) dy^j.$$

Concerning the left-hand side we have to remember also that $(\det J_1)\mu_1 = -dW^1$, since the driving system is of cofactor type with scKt J_1. It then readily follows, using the property (4.67), that the previous coordinate expression reduces to

$$\det(\bar{J}_2 + aP_2) \frac{\partial W^1}{\partial y^k} = \frac{\partial W(a)}{\partial y^k},$$

and this for all a. Making use of the expansion (4.64) we get

$$\frac{1}{\det(J_1)} \frac{\partial W^1}{\partial y^k} \sum_{i=1}^{n+1} \Delta_{(i)} a^{i-1} = \sum_{i=1}^{n+1} \frac{\partial W_{(i)}}{\partial y^k} a^{i-1}.$$

This implies that the functions $W_{(i)} - (\Delta_{(i)}/\det J_1)W^1$ do not depend on the y parameters, which together with (4.85) confirms our objective for the potential part in $H_{(i)}$.

It remains to look at the terms quadratic in the momenta. Expressed in the momenta associated to the u-variables, they become

$$\tfrac{1}{2} A^{ab}_{(i)} \frac{\partial u^c}{\partial x^a} \frac{\partial u^d}{\partial x^b} s_c s_d.$$

We know that in particular $A^{ab}_{(n)} = (\det J_1) g^{ab}$ (cf. (4.58)). Since $\bar{J}_2$ is a scKt with respect to g_2, it is symmetric in its covariant or contravariant representation. But $\bar{J}_2$ is diagonal in the u coordinates, hence the same is true for the transformed g_2. Let us rely here further on the fact that we proved by indirect means, mainly as a result of the simple Lemma 4.22, that $H_{(n)}$ will be time-independent when expressed in the (u, s) variables. The net conclusion then is that the diagonal elements of the transformed g_2 must be the product of $(\det J_1)^{-1}(y)$ with a function depending on the u variables only. It subsequently follows from the explicit expression of $A^{ab}_{(i)}$ in (4.83) and the fact that the ratios $\Delta_{(i)}/\det J_1$ are also functions of the u-variables only (see (4.85)), that the quadratic part of each $H_{(i)}$ will indeed become time-independent as well, when these functions are looked at as first integrals of the driven system along solutions of the driving system.

To complete the picture it is perhaps worth repeating that the final $H_{(1)}$ in that hierarchy is in fact the Hamiltonian H in the quasi-Hamiltonian representation of the full system. At last, if we act with $\bar{J}_2$ on the expression (4.81) for $i = 1$, we obtain a polynomial expression satisfied by $\bar{J}_2$

$$\begin{aligned}\bar{J}_2 A_{(1)2} &= \Delta_{(2)}\bar{J}_2 + \sum_{j=1}^{n-1}(-1)^j \Delta_{(j+2)}{\bar{J}_2}^{j+1} \\ &= \sum_{j=0}^{n-1}(-1)^j \Delta_{(j+2)}{\bar{J}_2}^{j+1}.\end{aligned}$$

We want to show that this is exactly the Cayley-Hamilton theorem applied to $\bar{J}_2$, as this would be comforting for the internal consistency of our results. Making use of (4.85) and the fact that $(\det J_1)^{-1}A_{(1)2}$ is the cofactor of $\bar{J}_2$, we get

$$(\det \bar{J}_2)I = \sum_{j=0}^{n-1}(-1)^j \sigma_{n-j-1}(u){\bar{J}_2}^{j+1}.$$

The coefficient of ${\bar{J}_2}^n$ on the right is now $(-1)^{n-1}$, so let us multiply by $(-1)^{n-1}$ to make this $+1$,

$$\begin{aligned}(-1)^{n-1}(\det \bar{J}_2)I &= \sum_{j=0}^{n-1}(-1)^{j+n-1}\sigma_{n-j-1}(u){\bar{J}_2}^{j+1} \\ &= \sum_{k=1}^{n}(-1)^{n+k}\sigma_{n-k}(u){\bar{J}_2}^{k}\end{aligned}$$

or thus, since $\det \bar{J}_2 = \sigma_n(u)$,

$$0 = \sum_{k=0}^{n}(-1)^{n+k}\sigma_{n-k}(u){\bar{J}_2}^{k}$$

and this expresses indeed that $\bar{J}_2$ satisfies its own characteristic equation.

4.8 Examples

In this section we will give some illuminating examples. The first example is inspired by an integrability study in [53] of so-called 'generalized Hénon-Heiles systems', which are systems of SODEs of the form

$$\begin{aligned}\ddot{q}_1 &= -c_1 q_1 + b q_1^2 - a q_2^2,\\ \ddot{q}_2 &= -c_2 q_2 - 2m q_1 q_2.\end{aligned}$$

Compared to older case studies of integrability of genuine Hénon-Heiles systems, the generalization comes from the extra parameters a and m, which are motivated by allowing a Lagrangian description of the system in which the Hessian of the Lagrangian need not be (a constant multiple of) the unit matrix. The investigation carried out in [53] mainly consisted in looking for all possible parameter cases for which the system has two independent quadratic first integrals. It led to the identification of three new cases, which are in some sense degenerate cases, because either a or m is zero, meaning that the Lagrangian one originally thought of is degenerate. As a result, there is no corresponding Hamiltonian which in the standard cases is always available as the first of two first integrals in involution. A subsidiary question then was: to what extent can the two first integrals in those degenerate cases be understood as being in involution, and it was argued that this question can be resolved in principle by constructing a suitably adapted non-standard Poisson structure. As the equations in the case that either a or m is zero, clearly exhibit partial decoupling, there is a good chance that those degenerate cases actually fit within our present theory, which is what we will discuss now.

Consider the case where $m = 0$ and $b = 0$, so that the system reduces to

$$\begin{aligned}\ddot{q}_1 &= -c_1 q_1 - a q_2^2,\\ \ddot{q}_2 &= -c_2 q_2.\end{aligned}$$

These equations are of course easy to solve without further ado, but they must serve here in the first place to illustrate various aspects of our theory. The second equation plays the role of driving equation. For consistency with the notations in the preceding sections we rename the variables as $q_2 = y$, $q_1 = x$ and write

$$\begin{aligned}\ddot{y} &= -c_2 y,\\ \ddot{x} &= -c_1 x - a y^2.\end{aligned}$$

Putting $p_y = \dot{y}$ and $p_x = \dot{x}$ the two quadratic integrals read

$$\begin{aligned} F_2 &= \tfrac{1}{2}p_y^2 + \tfrac{1}{2}c_2y^2 \\ F_1 &= \tfrac{1}{2}(c_1 - 4c_2)p_x^2 + 2ayp_xp_y - 2axp_y^2 + \tfrac{1}{2}c_1(c_1 - 4c_2)x^2 \\ &\qquad + a(c_1 - 2c_2)xy^2 + \tfrac{1}{2}a^2y^4. \end{aligned}$$

Obviously F_2 is a Hamiltonian for the driving equation and is likely to be identifiable with the function $H_{(2)}$ in the theory (see (4.30), knowing that $n = 1$ here). The idea now is the following. Since F_1 is a first integral of the complete system, its quadratic part identifies a Killing tensor A. Looking at the tensor J of which A is the cofactor, this may or may not be a scKt in general. It was shown in [40] however that this will always be the case for the Euclidean metric in dimension 2. If the component J_1 is nonsingular therefore, we must be in a situation covered by our present theory and all the features we discussed should apply, with $F_1 = H_{(1)} = H$, the Hamiltonian of the quasi-Hamiltonian representation (4.6). The extra assumption that the driven system should have a genuine potential, parametrically depending on the y-variables, is also automatically satisfied here by dimension.

From the expression of F_1, we see that

$$A = \begin{pmatrix} -4ax & 2ay \\ 2ay & c_1 - 4c_2 \end{pmatrix}. \tag{4.87}$$

and $A = \operatorname{cof} J$, where J is the tensor with the following matrix components

$$J = \begin{pmatrix} c_1 - 4c_2 & -2ay \\ -2ay & -4ax \end{pmatrix}. \tag{4.88}$$

This tensor is indeed a scKt with respect to the Euclidean metric, and its J_1 component is nonsingular (assuming $c_1 \neq 4c_2$), so we are in business and $A^1 = \operatorname{cof} J_1 = 1$. For completeness, observe that

$$\mu = -c_2ydy - (c_1x + ay^2)dx$$

in this example and that $A\mu = -dW$ indeed, with

$$W = \tfrac{1}{2}c_1(c_1 - 4c_2)x^2 + a(c_1 - 2c_2)xy^2 + \tfrac{1}{2}a^2y^4.$$

The function h for the standard Hamiltonian representation of the driven equation (with parameter y) is given by

$$h = \tfrac{1}{2}p_x^2 + ay^2x + \tfrac{1}{2}c_1x^2.$$

Let us now further illustrate the subtleties of the theory, as explained in Sections 4.4 to 4.6. We start by computing the function $\tilde{h}$ as defined in (4.57). The tensor $\bar{J}_2$ here reads

$$\bar{J}_2 = -\left(4ax + \frac{4a^2y^2}{c_1 - 4c_2}\right)\frac{\partial}{\partial x} \otimes dx. \tag{4.89}$$

and the new momenta defined in (4.54) and (4.55) become

$$\begin{aligned} p_y &= (c_1 - 4c_2)\tilde{p}_y, \\ p_x &= \tilde{p}_x - 2ay\tilde{p}_y. \end{aligned}$$

In those new variables, the expressions for F_1 and F_2 become

$$\begin{aligned} F_1 &= \tfrac{1}{2}(c_1 - 4c_2)\tilde{p}_x^2 - 2a^2y^2(c_1 - 4c_2)\tilde{p}_y^2 - 2ax(c_1 - 4c_2)^2\tilde{p}_y^2 \\ &\quad + \tfrac{1}{2}c_1(c_1 - 4c_2)x^2 + a(c_1 - 2c_2)xy^2 + \tfrac{1}{2}a^2y^4 \\ F_2 &= \tfrac{1}{2}(c_1 - 4c_2)^2\tilde{p}_y^2 + \tfrac{1}{2}c_2y^2. \end{aligned}$$

For F_1 this is in agreement with (4.65). For the computation of $\tilde{h}$, on the other hand, we need the generating function (4.56) of the time-dependent canonical transformation $(x, p_x) \leftrightarrow (x, \tilde{p}_x)$. We see from (4.88) that $J_2^1 = -2ay = \partial\psi/\partial x$ with $\psi = -2axy$. The generating function F thus reads

$$F(x, \tilde{p}_x, t) = x\tilde{p}_x - 2axy(t)\tilde{p}_y(t),$$

and computing its partial time derivative involves making use of the driving equation:

$$\frac{\partial F}{\partial t} = -2ax\dot{y}\tilde{p}_y - 2axy\dot{\tilde{p}}_y = -2ax(c_1 - 4c_2)\tilde{p}_y^2 - \frac{2axc_2y^2}{c_1 - 4c_2}.$$

The resulting expression for $\tilde{h}$ is found to be

$$\tilde{h} = \tfrac{1}{2}\tilde{p}_x^2 + 2a^2y^2\tilde{p}_y^2 - 2ax(c_1 - 4c_2)\tilde{p}_y^2 - 2ay\tilde{p}_x\tilde{p}_y + ay^2x + \tfrac{1}{2}c_1x^2 + \frac{2ay^2xc_2}{c_1 - 4c_2}.$$

One can verify that $\tilde{h}$ and $H_{(1)} = F_1$ indeed verify the requirement (4.60) of Lemma 4.19 to within an additive function of time, which is the function

$$4a^2y^2\tilde{p}_y^2 - \frac{a^2y^4}{2(c_1 - 4c_2)}$$

and of course can be ignored in writing down Hamilton's equations. For the final canonical transformation to be applied to $\tilde{h}$, we need the eigenfunction $u(y,x)$ of $\bar{J}_2$, which is found to be (see (4.89))

$$u(y,x) = -\frac{4a^2y^2}{c_1-4c_2} - 4ax.$$

The time-dependent canonical transformation with generating function $F(x,s,t) = s\,u(y(t),x)$ now transforms the relevant part of $\tilde{h}$ into the function

$$(\det J_1)^{-1}F_1 = 8a^2s^2 + \tfrac{1}{2}u(c_1-4c_2)\tilde{p}_y^2 + \tfrac{1}{2}u\frac{c_2}{c_1-4c_2}y^2 + \frac{c_1}{32a^2}u^2.$$

It then easily follows by taking into account that F_2 is a constant along solutions of the driving equation, that

$$F_1 = 8(c_1-4c_2)a^2s^2 + uF_2 + \frac{c_1(c_1-4c_2)}{32a^2}u^2,$$

which indeed no longer depends explicitly on time.

The case where $a=0$ and $b=-2m$ ($m\neq 0$) fits in a similar way within the present theory. But the other degenerate case where $a=0$, $b=-2m/5$ and $c_2=4c_1$ ($m\neq 0$) only fits into the present theory to some extent. In this case the system reduces to

$$\begin{aligned}\ddot{q}_1 &= -c_1q_1 - \tfrac{2m}{5}q_1^2,\\ \ddot{q}_2 &= -4c_1q_2 - 2mq_1q_2.\end{aligned}$$

Now the first equation plays the role of driving equation, so let us rename the variables as $q_1=y$, $q_2=x$ and write

$$\begin{aligned}\ddot{y} &= -c_1y - \tfrac{2m}{5}y^2,\\ \ddot{x} &= -4c_1x - 2myx.\end{aligned}$$

Putting $p_y=\dot{y}$ and $p_x=\dot{x}$, the two quadratic first integrals read

$$\begin{aligned}F_1 &= \tfrac{1}{2}p_y^2 + \tfrac{1}{2}c_1y^2 + \tfrac{2}{3}y^3,\\ F_2 &= yp_xp_y - xp_y^2 + c_1y^2x + \tfrac{2m}{5}xy^3.\end{aligned}$$

Clearly, F_1 is a first integral of the driving equation and F_2 is a first integral of the complete system. So the quadratic part of F_2 determines a Killing tensor A with matrix components

$$A = \begin{pmatrix} -2x & y \\ y & 0 \end{pmatrix}.$$

The 1-form

$$\mu = -(c_1 y + \tfrac{2m}{5} y^2)dy - (4c_1 x + 2myx)dx$$

indeed satisfies $A\mu = -dW$, with

$$W = c_1 y^2 x + \tfrac{2m}{5} xy^3.$$

This is equivalent with $D_J \mu = 0$ where J is the scKt of which A is the cofactor tensor,

$$J = \begin{pmatrix} 0 & -y \\ -y & -2x \end{pmatrix}.$$

Unfortunately the J_1 part of J is zero, so that we are not in the generic case of a nonsingular J_1, which was one of the basic assumptions in the preceding sections. But the two integrals can be understood by the fact that the system is a driven cofactor system. This also explains why the equations partially decouple.

With a second simple example, we want to illustrate and test mainly the beginning of our theory, before suitable coordinates for partial decoupling have been identified, i.e. the situation covered by the conditions of Definition 4.11. Consider the system

$$\begin{aligned} \ddot{q}_1 &= 5q_1 - 4q_2, \\ \ddot{q}_2 &= q_1. \end{aligned}$$

The corresponding SODE is

$$\Gamma = \dot{q}_1 \frac{\partial}{\partial q_1} + \dot{q}_2 \frac{\partial}{\partial q_2} + (5q_1 - 4q_2)\frac{\partial}{\partial \dot{q}_1} + q_1 \frac{\partial}{\partial \dot{q}_2},$$

so the connection coefficients are all zero and the $(1,1)$ tensor Φ has the following matrix representation

$$(\Phi^i_j) = \left(-\frac{\partial f^i}{\partial q^j}\right) = \begin{pmatrix} -5 & 4 \\ -1 & 0 \end{pmatrix}.$$

With such a simple, constant Jacobi endomorphism, finding a distribution K which entails submersiveness of the system, i.e. which satisfies the conditions (4.9) of Definition 4.11, is simply a matter of looking for a 1-dimensional eigenspace of Φ. We choose

$$K = \mathrm{sp}\left\{\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right\}. \tag{4.90}$$

Note that, as an alternative for the distribution K just defined, we could also consider the other eigenspace of Φ, spanned by

$$\left\{4\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right\}.$$

But then in the adapted coordinates, it so happens that the J_1 part of J is zero and in this situation one of the basic assumptions of our theory is not satisfied. So let us consider K as defined by (4.90). Submersiveness now is ensured and it is not related to the existence of a cofactor representation of our system. The problem of detecting such a representation is quite interesting in its own right. To some extent it bears resemblance to the inverse problem of the calculus of variations, because there is a certain freedom in selecting a multiplier matrix g first. For example, making the obvious choice of the unit matrix for g or, expressed in more mechanical terms, associating the given equations with the standard kinetic energy $T = \frac{1}{2}(\dot q_1^2 + \dot q_2^2)$, will do the job, in contrast to what we previously claimed in [55]. Indeed, one can verify that with this g, the possible special conformal Killing tensor J for which the right-hand sides of the equations will satisfy the condition $D_J\mu = 0$ is given by

$$J = \begin{pmatrix} 8q_1 & q_1 + 4q_2 \\ q_1 + 4q_2 & 2q_2 \end{pmatrix}$$

with, in this case, $\mu = (5q_1 - 4q_2)dq_1 + 2q_2 dq_2$. We can now compute the orthogonal complement of the distribution K and see whether the final conditions for a driven cofactor system are verified. It follows that

$$K^\perp = \mathrm{sp}\left\{\frac{\partial}{\partial q_1} - \frac{\partial}{\partial q_2}\right\}.$$

We have

$$\mathrm{D}^H\mu = 5dq_1 \otimes dq_1 + dq_1 \otimes dq_2 - 4dq_2 \otimes dq_1.$$

It is clear then that $\mathrm{D}^H\mu(K^\perp, K) \neq 0$, whereas obviously $d\mu(K,K) = 0$ by dimension. Hence, all requirements of Definition 4.11 are met. But note that, as we discussed in [55], the following choice for g

$$(g_{\alpha\beta}) = \begin{pmatrix} 1 & -1 \\ -1 & 10 \end{pmatrix}$$

turns out to be appropriate as well. Then, a scKt J with respect to g such that $D_J\mu = 0$ is found to be

$$(J_{\alpha\beta}) = \begin{pmatrix} 2q_1 - 2q_2 & q_1 + 8q_2 \\ q_1 + 8q_2 & -4q_1 + 40q_2 \end{pmatrix}.$$

Multiplying the right-hand sides of the equations with g, the 1-form μ is now found to be

$$\mu = 4(q_1 - q_2)dq_1 + (5q_1 + 4q_2)dq_2,$$

and computing the cofactor tensor A of J, one can verify that $A\mu$ indeed is closed (or equivalently $D_J\mu = 0$). Since $g_{11} + g_{21} = 0$, it easily follows that

$$K^\perp = \mathrm{sp}\left\{\frac{\partial}{\partial q_1}\right\}.$$

We have

$$\mathrm{D}^H\mu = 4dq_1 \otimes dq_1 + 5dq_1 \otimes dq_2 - 4dq_2 \otimes dq_1 + 4dq_2 \otimes dq_2.$$

It is clear then that $\mathrm{D}^H\mu(K^\perp, K) \neq 0$, whereas $d\mu(K,K) = 0$, again by dimension. So, also in this case all requirements of Definition 4.11 are met. This means that the given system in fact has two independent cofactor representations.

To end this section, let us verify all the other features of our theory for this second cofactor representation. Integrating the distributions K and $K^\perp$, suitable coordinates for the decoupling are found to be $x = q_2$, $y = q_1 - q_2$ and the transformed system becomes

$$\begin{aligned} \ddot{y} &= 4y, \\ \ddot{x} &= y + x. \end{aligned}$$

In these coordinates we have

$$(g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$

and

$$\left(J^{\alpha}_{\beta}\right) = \begin{pmatrix} 2y & 3y+9x \\ \frac{1}{3}y + x & 6x \end{pmatrix}. \tag{4.91}$$

The function h for the standard Hamiltonian representation of the driven equation (with parameter y) is given by

$$h = \tfrac{1}{18}p_x^2 - 9xy - \tfrac{9}{2}x^2.$$

The first integral of the driving equation (see (4.30)) reads

$$H_{(2)} = \tfrac{1}{2}p_y^2 - 2y^2.$$

For the first integral H (see (4.6)) we need the function W which is determined by $A\mu = -dW$. We find

$$W = 9x^2y - 6xy^2 + y^3$$

and

$$H = 3xp_y^2 - \left(\tfrac{1}{3}y + x\right)p_x p_y + \frac{y}{9}p_x^2 + 9x^2y - 6xy^2 + y^3.$$

We now compute the function $\tilde{h}$ as defined in (4.57). The tensor $\bar{J}_2$ here reads

$$\bar{J}_2 = \left(-\frac{9x^2}{2y} - \tfrac{1}{2}y + 3x\right)\frac{\partial}{\partial x} \otimes dx. \tag{4.92}$$

and the new momenta defined in (4.54) and (4.55) become

$$\begin{aligned} p_y &= 2y\tilde{p}_y, \\ p_x &= \tilde{p}_x + (3y+9x)\tilde{p}_y. \end{aligned}$$

In those new variables, the expressions for H and $H_{(2)}$ become

$$\begin{aligned} H &= \frac{y}{9}\tilde{p}_x^2 + \left(6xy^2 - y^3 - 9x^2y\right)\tilde{p}_y^2 + 9x^2y - 6xy^2 + y^3 \\ H_{(2)} &= 2y^2\tilde{p}_y^2 - 2y^2. \end{aligned}$$

For $H = H_{(n)} = H_{(1)}$ this is in agreement with (4.65). For the computation of $\tilde{h}$ on the other hand, we need the generating function (4.56) of the time-dependent canonical transformation $(x, p_x) \leftrightarrow (x, \tilde{p}_x)$. We see from (4.91) that $J_2^1 = 3y + 9x = \partial\psi/\partial x$ with $\psi = 3yx + \frac{9}{2}x^2$. The generating function F thus reads

$$F(x, \tilde{p}_x, t) = x\tilde{p}_x + \left(3yx + \tfrac{9}{2}x^2\right)\tilde{p}_y(t),$$

and for its partial time derivative (making use of the driving equation) we get

$$\begin{aligned}\frac{\partial F}{\partial t} &= 3x\dot{y}\tilde{p}_y + \left(3yx + \tfrac{9}{2}x^2\right)\dot{\tilde{p}}_y \\ &= -9x^2\tilde{p}_y^2 + 9x^2 + 6xy.\end{aligned}$$

This way we find the following expression for $\tilde{h}$

$$\tilde{h} = \frac{1}{18}\tilde{p}_x^2 + \left(3xy + \tfrac{1}{2}y^2 - \frac{9}{2}x^2\right)\tilde{p}_y^2 + \frac{1}{9}(9x + 3y)\tilde{p}_x\tilde{p}_y + \frac{9}{2}x^2 - 3xy.$$

So $\tilde{h}$ and H indeed satisfy (4.60) of Lemma 4.19 to within the additive function of time $y^2\tilde{p}_y^2 - \frac{1}{2}y^2$. For the final canonical transformation we need the eigenfunction $u(y, x)$ of $\bar{J}_2$,

$$u(y, x) = -\frac{9x^2}{2y} - \tfrac{1}{2}y + 3x.$$

The time-dependent canonical transformation with generating function $F(x, s, t) = s\,u(y(t), x)$ now transforms the relevant part of $\tilde{h}$ into the function

$$(\det J_1)^{-1}H = -\frac{u}{y}s^2 - uy + uy\tilde{p}_y^2.$$

Making use of $H_{(2)}$ it then easily follows that

$$H = -2us^2 + uH_{(2)},$$

which indeed no longer depends explicitly on time.

NEDERLANDSTALIGE SAMENVATTING

In het kort kan het hoofddoel van deze scriptie als volgt in één zin omschreven worden: het is onze intentie een intrinsieke of coördinaatonafhankelijke beschrijving te geven van een aantal resultaten omtrent Hamiltoniaanse systemen, die rechtstreeks of onrechtstreeks te maken hebben met tijdsafhankelijke systemen waarvan de corresponderende Hamilton-Jacobivergelijking kan opgelost worden via scheiden van de veranderlijken.

In het eerste hoofdstuk schetsen we de wiskundige achtergrond van deze scriptie. We starten met een herhaling van de basisdefinities omtrent Poissonvariëteiten. Daarna bestuderen we in Paragraaf 1.2 afleidingsoperatoren. Enerzijds vermelden we een aantal definities en stellingen uit de klassieke theorie van Frölicher en Nijenhuis. Anderzijds overlopen we de voornaamste ingrediënten van de calculus langs de raakbundelprojectie $\tau : TM \to M$ en bespreken we de theorie rond afleidingsoperatoren van differentiaalvormen langs τ. Centraal in dit inleidende hoofdstuk staat echter de bespreking van drie belangrijke concepten: Poisson-Nijenhuisvariëteiten, 'speciale conforme Killingtensoren' en de Hamilton-Jacobivergelijking.

In Paragraaf 1.3 introduceren we Poisson-Nijenhuisvariëteiten. Een Poisson-Nijenhuisvariëteit bestaat uit een Poissontensor P en een (1,1)-tensorveld R waarvoor geldt dat $RP = PR$ een tweede compatibel Poissonhaakje definieert. Een dergelijke variëteit heeft dus een bi-Hamiltoniaanse structuur en het tensorveld R wordt dan de recursieoperator genoemd. Uit de compatibiliteit van de twee Poissonhaakjes volgt dat de Nijenhuistorsie van R verdwijnt. Interessant is dat, wanneer de recursieoperator in elk punt n verschillende eigenwaarden heeft, er zogenoemde Darboux-Nijenhuiscoördinaten gedefinieerd kunnen worden op de Poisson-Nijenhuisvariëteit. Deze coördinaten zijn in feite Darbouxcoördinaten voor de Poissontensor en brengen bovendien de recursieoperator in diagonale vorm.

Een bekend voorbeeld van een Poisson-Nijenhuisstructuur op de co-raakbundel ontstaat als volgt. Beschouw een (1,1)-tensorveld J op een Riemannvariëteit. Men kan aantonen dat als de Nijenhuistorsie van J verdwijnt, deze van de complete lift $\tilde{J}$ van J naar de co-raakbundel ook verdwijnt en dan definiëren $\tilde{J}$ en de canonische symplectische structuur ω een Poisson-Nijenhuisstructuur op de co-raakbundel. Het tweede Poissonhaakje is dan van de vorm

$$\{q_i, q_j\}_1 = 0, \qquad \{q_i, p_j\}_1 = -J^i_j, \qquad \{p_i, p_j\}_1 = p_k \left(\frac{\partial J^k_j}{\partial q^i} - \frac{\partial J^k_i}{\partial q^j} \right)$$

en is compatibel met het Poissonhaakje bepaalt door ω.

Wanneer we in Hoofdstuk 4 gestuurde cofactorsystemen in detail bespreken, spelen speciale conforme Killingtensoren een belangrijke rol. Het zijn conforme Killingtensoren met een specifieke vorm. Ze moeten namelijk t.o.v. de beschouwde Riemannmetriek g voldoen aan relaties van de vorm

$$J_{ij|k} = \tfrac{1}{2}(\alpha_i g_{jk} + \alpha_j g_{ik})$$

met $|k$ de covariante afgeleide en $\alpha_i = \partial(\operatorname{tr} J)/\partial q^i$. Als een dergelijk tensorveld enkelvoudige eigenwaarden heeft, wordt het ook wel een Benentitensor genoemd, naar S. Benenti die deze klasse van tensorvelden als eerste bestudeerde. In Paragraaf 1.4 herhalen we de definitie en enkele interessante eigenschappen. Zo is bijvoorbeeld de cofactortensor van een niet-singuliere speciale conforme Killingtensor een Killingtensor, m.a.w. hij bepaalt een kwadratische eerste integraal van het systeem. Verder geldt er dat de Nijenhuistorsie van een speciale conforme Killingtensor automatisch verdwijnt. Een ander onderzoeksdomein waarin speciale conforme Killingtensoren opduiken is de studie van de Hamilton-Jacobivergelijking, meer bepaald het scheiden van de veranderlijken.

Voor een Hamiltoniaan $H(t,q,p)$ is de Hamilton-Jacobivergelijking

$$H\left(t,q,\frac{\partial S}{\partial q}\right)+\frac{\partial S}{\partial t}=0$$

een partiële differentiaalvergelijking voor $S(t,q)$. Meestal is het niet evident om een oplossing te vinden, maar in bepaalde gevallen bestaat er een efficiënte oplossingsmethode. Het idee is dan dat er een complete oplossing van de vorm

$$S(t,q,\alpha)=S_0(t,\alpha)+\sum_{i=1}^{n}S_i(q^i,\alpha)$$

gezocht wordt. Als zo'n oplossing bestaat, dan zegt men dat de Hamiltoniaan scheidbaar is en dat de geassocieerde Hamilton-Jacobivergelijking opgelost kan worden via scheiden van de veranderlijken. De $S_i(q^i,\alpha)$ kunnen dan in principe eenvoudig bepaald worden aan de hand van kwadraturen. Omdat deze oplossingsmethode zo interessant is, is het scheiden van de veranderlijken sinds het ontstaan van de Hamilton-Jacobivergelijking een levendig onderzoeksdomein. Het idee is enerzijds om Hamiltonianen te identificeren die scheidbaar zijn in gegeven coördinaten en anderzijds om coördinaten te vinden waarin een gegeven Hamiltoniaan scheidbaar is.

In Paragraaf 1.5 bespreken we voor autonome Hamiltoniaanse systemen kort de belangrijkste resultaten. De eerste coördinaatafhankelijke voorwaarden, deze van Liouville en Stäckel, zijn specifiek voor Hamiltonianen die scheidbaar zijn in orthogonale coördinaten. In 1904 leidde Levi-Civita echter een test af voor een willekeurig Hamiltoniaans systeem. Hij stelde dat een algemene Hamiltoniaan $H(q,p)$ scheidbaar is in de coördinaten (q^i,p_i) als en slechts als

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial^2 H}{\partial q^i\,\partial q^j}\frac{\partial H}{\partial p_j}-\frac{\partial^2 H}{\partial q^i\,\partial p_j}\frac{\partial H}{\partial q^j}\right)=\frac{\partial H}{\partial q^i}\left(\frac{\partial^2 H}{\partial p_i\,\partial q^j}\frac{\partial H}{\partial p_j}-\frac{\partial^2 H}{\partial p_i\,\partial p_j}\frac{\partial H}{\partial q^j}\right),$$

waarbij er geen sommatie is over herhaalde indices en $i,j=1,\dots,n$ met $i\neq j$. Deze resultaten hebben een groot nadeel, je kan enkel nagaan of de Hamilton-Jacobivergelijking scheidbaar is in de gegeven coördinaten. Je moet dus het geluk hebben dat je reeds in de scheidingscoördinaten werkt opdat je een positief resultaat zou hebben. Ze vertellen je ook niet of er (andere) scheidingscoördinaten bestaan en hoe je die dan kan construeren. En dit is net het grote voordeel van de intrinsieke, i.e. coördinaatonafhankelijke resultaten die we beschrijven in Paragraaf 1.5.2. We hebben in het bijzonder aandacht voor de resultaten van Eisenhart en Benenti. Zoals reeds vermeld, spelen speciale conforme Killingtensoren ook hier een belangrijke rol: als er voor een gegeven Riemannmetriek g een speciale conforme Killingtensor bestaat met functioneel onafhankelijke eigenfuncties λ_i, dan is de corresponderende Hamiltoniaan $H = \frac{1}{2}g^{ij}p_ip_j$ scheidbaar in de orthogonale coördinaten λ_i.

In Paragraaf 1.5 ligt de focus op autonome Hamiltoniaanse systemen. Voor tijdsafhankelijke systemen zijn er heel wat minder resultaten bekend. Er bestaat echter wel een veralgemening van de Levi-Civitavoorwaarden, opgesteld door Forbat in 1944, en verderop steeds Forbats voorwaarden genoemd. Forbat toonde aan dat een tijdsafhankelijke Hamiltoniaan $H(t, q, p)$ scheidbaar is in de coördinaten (t, q^i, p_i) als en slechts als

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial^2 H}{\partial q^i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial q^i\,\partial p_j}\frac{\partial H}{\partial q^j}\right) = \frac{\partial H}{\partial q^i}\left(\frac{\partial^2 H}{\partial p_i\,\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial^2 H}{\partial p_i\,\partial p_j}\frac{\partial H}{\partial q^j}\right),$$

$$\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial q^i\,\partial t} = \frac{\partial H}{\partial q^i}\frac{\partial^2 H}{\partial p_i\,\partial t},$$

waarbij er terug geen sommatie is over herhaalde indices en $i, j = 1, \ldots, n$ met $i \neq j$. Voor tijdsonafhankelijke Hamiltoniaanse systemen reduceren deze voorwaarden tot de Levi-Civitavoorwaarden. En net zoals bij de Levi-Civitavoorwaarden kunnen we ook hier de opmerking maken dat je het geluk moet hebben reeds scheidingscoördinaten te hebben gekozen. Maar in tegenstelling tot de resultaten voor de Levi-Civitavoorwaarden bestaat er nog geen coördinaatonafhankelijke formulering van Forbats voorwaarden. Een eerste doel van ons onderzoek is dan ook een intrinsieke beschrijving geven van Forbats voorwaarden. We willen een test vinden voor het bestaan van scheidingscoördinaten die in een willekeurig coördinatensysteem kan uitgevoerd worden en die bovendien vertelt hoe de scheidingscoördinaten dan geconstrueerd kunnen worden. Merk op dat we ons beperken tot de klassieke benadering van de Hamilton-Jacobitheorie waarbij de scheidingscoördinaten bepaald worden aan de hand van de originele coördinaten via een punttransformatie, i.e. $Q^i = Q^i(t, q), P_i = p_k\partial q^k/\partial Q^i$.

Voor de beschrijving van tijdsafhankelijke Hamiltoniaanse systemen wordt in de literatuur meestal $\mathbb{R} \times T^*M$ als model voor de beschouwde ruimte vooropgesteld. Dit is op zich geen probleem, maar als men denkt aan toepassingen waar ons onderzoek in kadert, dan is het evident dat men tijdsafhankelijke coördinatentransformaties wil toelaten, d.w.z. transformaties van de vorm $(t = t, Q^i = Q^i(t, q))$ waarbij (t, q^i)

coördinaten op $\mathbb{R} \times M$ zijn. En dergelijke transformaties respecteren de productstructuur van $\mathbb{R} \times M$ niet! Het, voor ons, betere model voor de beschrijving van tijdsafhankelijke systemen zal dan ook onstaan als we vertrekken van een bundel over $\mathbb{R}$, $\tau : E \to \mathbb{R}$ met $\dim E = n+1$. Lokale coördinaten op E zullen we aanduiden met (t, q^i). De eerste-orde jetbundel $J^1\tau$ is dan een affiene bundel over E met geïnduceerde coördinaten $(t, q^i, \dot{q}^i)$. Voor de beschrijving van tijdsafhankelijke Lagrangiaanse systemen is dit een goed model. Wij werken echter met tijdsafhankelijke Hamiltoniaanse systemen en hiervoor is de duale bundel van de eerste-orde jetbundel, genoteerd als $J^1\tau^*$, het meest geschikt. In Paragraaf 2.1 bespreken we de constructie uitgebreid, maar in feite is $J^1\tau^*$ de quotiëntruimte $T^*E/\langle dt \rangle$. Ieder punt $m \in J^1\tau^*$ is dan een equivalentieklasse van covectoren $\langle \alpha \rangle \bmod dt$ in $\pi(m)$ en m heeft coördinaten (t, q^i, p_i) als $\alpha_{(t,q)} = p_i dq^i \bmod dt$. Een Hamiltoniaan is een sectie h van de bundel $\rho : T^*E \to J^1\tau^*$. Lokaal definieert h een functie H op $J^1\tau^*$ als volgt, $h : (t, q, p) \mapsto (t, q, p_0 = -H(t, q, p), p)$. Het geassocieerd Hamiltoniaans systeem is dan een vectorveld X_h op $J^1\tau^*$ dat voldoet aan

$$i_{X_h} h^*\omega_E = 0 \quad \text{and} \quad \langle X_h, dt \rangle = 1,$$

waarbij ω_E de canonische symplectische vorm is op T^*E, zodat lokaal $h^*\omega_E = dp_i \wedge dq^i - dH \wedge dt$. In coördinaten is X_h van de vorm

$$X_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

In de studie en karakterisatie van Hamilton-Jacobi scheidbaarheid spelen Poisson-Nijenhuisstructuren en de geassocieerde Darboux-Nijenhuiscoördinaten een belangrijke rol. Zo is in de context van de standaard Hamilton-Jacobitheorie voor tijdsonafhankelijke Hamiltoniaanse systemen, de recursieoperator van een Poisson-Nijenhuisstructuur meestal de complete lift naar de co-raakbundel T^*M van een (1,1)-tensorveld op M. Dit diende als inspiratiebron voor onze intrinsieke beschrijving van Forbats voorwaarden. Vandaar dat we in Hoofdstuk 2 eerst van naderbij bekijken hoe de theorie rond het liften van geometrische objecten naar de co-raakbundel uitgebreid kan worden naar $J^1\tau^*$. Het hoofddoel is het intrinsiek definiëren van de complete lift naar $J^1\tau^*$ van een (1,1)-tensorveld op E en het begrijpen van de daaraan gerelateerde eigenschappen.

In Paragraaf 2.3 herhalen we eerst de belangrijkste liftoperaties van een geometrisch object op een variëteit M naar zijn co-raakbundel T^*M. De nieuwe bijdragen starten in Paragraaf 2.4 met de bespreking van verschillende liften van vectorvelden en 1-vormen naar $J^1\tau^*$. We definiëren de verticale lift van een 1-vorm $\alpha = \alpha_0(t,q)dt + \alpha_i(t,q)dq^i$ op E, als het vectorveld $\alpha^v = \alpha_i \partial/\partial p_i \in \mathcal{X}(J^1\tau^*)$. Verder definiëren we de complete lift van twee klassen van vectorvelden op E, namelijk het $C^\infty(E)$-moduul van verticale vectorvelden op E, $\mathcal{X}_V(E)$, en de verzameling van vectorvelden met de eigenschap $\langle X, dt\rangle = 1$ genoteerd als $\mathcal{X}_t(E)$. Voor alle $X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E)$ is de complete lift het vectorveld $\widetilde{X} \in \mathcal{X}(J^1\tau^*)$ dat voldoet aan de volgende voorwaarden: $\widetilde{X}$ is π-verbonden met X, waarbij π de natuurlijke projectie $\pi : J^1\tau^* \to E$ is, en $\mathcal{L}_{\widetilde{X}}\Theta = 0$, met $\Theta = p_i dq^i \wedge dt$ de canonisch gedefinieerde 2-vorm op $J^1\tau^*$. In coördinaten,

$$
\begin{aligned}
\widetilde{X} &= X^i \frac{\partial}{\partial q^i} - p_j \frac{\partial X^j}{\partial q^i} \frac{\partial}{\partial p_i}, &&\text{voor } X = X^i(t,q)\frac{\partial}{\partial q^i} \in \mathcal{X}_V(E)\\
\widetilde{X} &= \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} - p_j \frac{\partial X^j}{\partial q^i} \frac{\partial}{\partial p_i}, &&\text{voor } X = \frac{\partial}{\partial t} + X^i(t,q)\frac{\partial}{\partial q^i} \in \mathcal{X}_t(E).
\end{aligned}
$$

De vectorvelden α^v en $\widetilde{X}$ bepalen samen een lokale basis voor de vectorvelden op $J^1\tau^*$. Dit is de belangrijkste reden voor hun introductie want zo zijn ze ideaal om gebruik van te maken wanneer we op een coördinaatonafhankelijke wijze andere geometrische objecten willen definiëren op $J^1\tau^*$.

Zo voeren we in Paragraaf 2.5 verschillende manieren in om een (1,1)-tensorveld op E te liften naar $J^1\tau^*$, met als resultaat respectievelijk een vectorveld, een 1-vorm of een (1,1)-tensorveld. Merk op dat we steeds een (1,1)-tensorveld R op E beschouwen met de eigenschap dat $R(dt) = 0$. Zo is $R_{\pi(m)}(m)$ goed gedefinieerd voor alle $m \in J^1\tau^*$ want zoals eerder gezegd is m geen co-vector in $\pi(m)$ maar een equivalentieklasse van covectoren $\bmod dt$. De belangrijkste lift van een (1,1)-tensorveld R op E is natuurlijk de complete lift, verder genoteerd als $\widetilde{R}$. Dit unieke (1,1)-tensorveld $\widetilde{R}$ op $J^1\tau^*$ wordt gedefinieerd door

$$
\begin{aligned}
&\widetilde{R}(\alpha^v) = R(\alpha)^v, \quad \forall\, \alpha \in \mathcal{X}^*(E)\\
&\widetilde{R}(\widetilde{X}) = \widetilde{R(X)} + (\mathcal{L}_X R)^v, \quad \forall\, X \in \mathcal{X}_V(E) \cup \mathcal{X}_t(E).
\end{aligned}
$$

In coördinaten bekomen we, voor

$$
R = R^i_j \frac{\partial}{\partial q^i} \otimes dq^j + R^i_0 \frac{\partial}{\partial q^i} \otimes dt,
$$

dat de complete lift gegeven wordt door

$$\widetilde{R} = R^i_j \left(\frac{\partial}{\partial q^i} \otimes dq^j + \frac{\partial}{\partial p_j} \otimes dp_i \right) + R^i_0 \frac{\partial}{\partial q^i} \otimes dt$$
$$+ p_i \left(\frac{\partial R^i_j}{\partial q^k} - \frac{\partial R^i_k}{\partial q^j} \right) \frac{\partial}{\partial p_j} \otimes dq^k + p_i \left(\frac{\partial R^i_k}{\partial t} - \frac{\partial R^i_0}{\partial q^k} \right) \frac{\partial}{\partial p_k} \otimes dt.$$

We bespreken heel wat interessante eigenschappen. Voor ons verder onderzoek is het onder meer van belang dat er een verband bestaat tussen de complete lift van een (1,1)-tensorveld R op E naar de co-raakbundel T^*E, om verwarring te vermijden genoteerd als $\widetilde{R}_{T^*}$, en de complete lift van hetzelfde tensorveld naar $J^1\tau^*$. Ze zijn namelijk ρ-verbonden. Dit betekent dat $\widetilde{R}_{T^*}(Y)$ ρ-verbonden is met $\widetilde{R}(Z)$ voor alle paren van ρ-verbonden vectorvelden $(Y, Z) \in \mathcal{X}(T^*E) \times \mathcal{X}(J^1\tau^*)$, met ρ de projectie $\rho : T^*E \to J^1\tau^*$. Daarnaast bewijzen we ook dat de Nijenhuistorsie van de complete lift $\widetilde{R}$ verdwijnt als en slechts als de Nijenhuistorsie van R verdwijnt. Dit resultaat is in het bijzonder belangrijk voor de bepaling van een Poisson-Nijenhuisstructuur op $J^1\tau^*$. We bespreken dit uitvoerig in Paragraaf 2.6. We tonen aan dat de complete lift $\widetilde{R}$ en de canonische Poissonafbeelding op $J^1\tau^*$ een Poisson-Nijenhuisstructuur definiëren als en slechts als de Nijenhuistorsie van R verdwijnt. Onder deze voorwaarde definieert ook de complete lift van R naar T^*E, $\widetilde{R}_{T^*}$, een Poisson-Nijenhuisstructuur op T^*E, bepaald door $\widetilde{R}_{T^*}$ en de Poissonafbeelding corresponderend met de canonische symplectische vorm ω_E op T^*E. Als we veronderstellen dat R algebraïsch diagonaliseerbaar is met verschillende eigenwaarden, dan bestaat er een lokale coördinatentransformatie op E die Darboux-Nijenhuiscoördinaten induceert voor beide Poisson-Nijenhuisstructuren, respectievelijk op $J^1\tau^*$ en T^*E. We bespreken deze constructie in detail in Paragraaf 2.6.1.

We maken nu hiervan gebruik om in Hoofdstuk 3 een coördinaatonafhankelijke beschrijving te geven van Forbats voorwaarden. Zoals eerder gezegd is $J^1\tau^*$ de geschikte ruimte om tijdsafhankelijke Hamiltoniaanse systemen te beschrijven. De hoop is dan ook om een intrinsieke voorwaarde voor scheidbaarheid van de Hamiltoniaan direct op $J^1\tau^*$ te formuleren. Maar we moeten voorzichtig zijn: bij een tijdsafhankelijke coördinatentransformatie krijgt de Hamiltoniaan extra termen, komende van de geïnduceerde transformatie van p_0 op T^*E. Daarom zal naast $J^1\tau^*$ ook de co-raakbundel T^*E een belangrijke rol spelen. Kort kunnen we stellen dat het een wisselwerking is tussen de complete lift van R naar T^*E en $J^1\tau^*$ die zal zorgen voor gerelateerde distributies op beide variëteiten. En het is de integrabiliteit van deze distributies, wat een coördinaatonafhankelijke voorwaarde is, die in geschikte coördinaten Forbats voorwaarden oplevert.

We starten in Paragraaf 3.2 met het definiëren van een distributie $\mathcal{D}_F$ geassocieerd aan een functie F op de co-raakbundel T^*E,

$$\mathcal{D}_F = \mathrm{sp}\,\{dF, \widetilde{R}_{T^*}(dF), \widetilde{R}^2_{T^*}(dF), \ldots, \widetilde{R}^n_{T^*}(dF)\}^\circ,$$

met $\widetilde{R}_{T^*}$ de complete lift naar T^*E van een (1,1)-tensorveld R op E waarvoor $R(dt) = 0$ en dat bovendien verondersteld wordt algebraïsch diagonaliseerbaar te zijn met verschillende eigenwaarden. Als de 1-vormen die $\mathcal{D}_F$ definiëren lineair onafhankelijk zijn, heeft $\mathcal{D}_F$ dimensie $n+1$. Er volgt dan dat $\mathcal{D}_F$ Lagrangiaans is en gelijk aan

$$\mathcal{D}_F^\perp = \mathrm{sp}\,\{X_F, \widetilde{R}_{T^*}(X_F), \ldots, \widetilde{R}^n_{T^*}(X_F)\}.$$

In dit geval bewijzen we dat als de Nijenhuistorsie van R verdwijnt, $\mathcal{D}_F$ integreerbaar is als en slechts als $dd_{\widetilde{R}_{T^*}}F\big|_{\mathcal{D}_F} = 0$. Op het einde van de paragraaf illustreren we het verband tussen de integreerbaarheid van $\mathcal{D}_F$ en de klassieke Hamilton-Jacobivergelijking voor de Hamiltoniaan F op T^*E. Maar dit is in feite de Hamilton-Jacobivergelijking voor een autonome Hamiltoniaan waarbij een van de variabelen toevallig t genoemd is en dit is natuurlijk niet waar we naar op zoek zijn.

Onze aandacht gaat naar het specifiek geval dat $F = \widetilde{H} := p_0 + H(t, q^i, p_i)$, en $\widetilde{H} = 0$ een sectie definieert van $\rho : T^*E \to J^1\tau^*$ en dus een tijdsafhankelijk Hamiltoniaans systeem op $J^1\tau^*$. In dit geval kunnen we een corresponderende distributie $\mathcal{D}_h$ definiëren op $J^1\tau^*$, namelijk

$$\mathcal{D}_h = \mathrm{sp}\,\{X_h, \widetilde{R}(X_h), \widetilde{R}^2(X_h), \ldots, \widetilde{R}^n(X_h)\},$$

waarbij $\widetilde{R}$ nu de complete lift is van R naar $J^1\tau^*$. In Paragraaf 3.3 bekijken we de interactie tussen de distributies $\mathcal{D}_{\widetilde{H}}$ en $\mathcal{D}_h$. Zo blijken ze niet alleen ρ-verbonden te zijn maar ook h-verbonden. Dit laatste is essentieel om aan te tonen dat $\mathcal{D}_h$ een integreerbare distributie is op $J^1\tau^*$ als en slechts als $\mathcal{D}_{\widetilde{H}}$ een integreerbare distributie is op T^*E. Hieruit volgt dat we de integreerbaarheid van $\mathcal{D}_h$ kunnen reduceren tot de voorwaarde $dd_{\widetilde{R}_{T^*}}\widetilde{H}\big|_{\mathcal{D}_{\widetilde{H}}} = 0$ op T^*E. Hier zijn we echter nog niet helemaal tevreden mee, het zou immers nog meer voldoening geven mochten we een voorwaarde voor de integreerbaarheid van $\mathcal{D}_h$ kunnen vinden direct op $J^1\tau^*$. Het h-verbonden zijn van de vectorvelden $\widetilde{R}^k_{T^*}(X_{\widetilde{H}})$ en $\widetilde{R}^k(X_h)$ speelt hierin opnieuw een belangrijke rol. Uiteindelijk vinden we dat $\mathcal{D}_h$ integreerbaar is als en slechts als $\mathcal{L}_{X_h}\omega_R|_{\mathcal{D}_h} = 0$. Hierbij is ω_R de 2-vorm op $J^1\tau^*$ gedefinieerd door $\omega_R = h^*\tau_R^* d\theta_E$. We beweren dat dit een intrinsieke versie is van Forbats voorwaarden. Met andere woorden als we voor een Hamiltoniaan H een (1,1)-tensorveld R vinden zodat aan de voorwaarde voldaan is, dan zullen er scheidingscoördinaten bestaan voor H. De vraag is dan nog hoe vinden we die? In Paragraaf 3.4 illustreren we dat de scheidingscoördinaten gegeven worden door de Darboux-Nijenhuiscoördinaten geassocieerd aan de beschouwde Poisson-Nijenhuisstructuur. Of dus dat H voldoet aan Forbats voorwaarden in die Darboux-Nijenhuiscoördinaten. We eindigen het hoofdstuk met een aantal voorbeelden in Paragraaf 3.5. We maken een zekere ansatz voor een Hamiltoniaan H en een (1,1)-tensorveld R op E en gaan binnen deze klasse op zoek naar mogelijke Hamiltonianen en een corresponderend (1,1)-tensorveld dat aan onze voorwaarde voldoen.

In het tweede deel van deze scriptie beschouwen we een specifieke klasse van gedeeltelijk ontkoppelde tweede-orde differentiaalvergelijkingen, zogenaamde gestuurde cofactorsystemen. Deze systemen werden geïntroduceerd en uitgebreid geanalyseerd op de Euclidische ruimte. Later werden ze gedeeltelijk veralgemeend naar Riemannvariëteiten. Het tweede doel van ons onderzoek gaat naar het vervolledigen van deze veralgemening.

Op een Riemannvariëteit zijn gestuurde cofactorsystemen differentiaalvergelijkingen van de vorm

$$\begin{aligned} \ddot{y}^i &= -\Gamma^i_{jk}(y)\dot{y}^j\dot{y}^k + Q^i(y), && i = 1,\dots,m \\ \ddot{x}^a &= -\Gamma^a_{bc}(x)\dot{x}^b\dot{x}^c + Q^a(y,x) && a = 1,\dots,n, \quad (\text{here } n+m = \dim M). \end{aligned}$$

Deze vergelijkingen zijn duidelijk gedeeltelijk ontkoppeld, het y-systeem wordt het sturend systeem genoemd en de overige vergelijkingen in x zijn het gestuurd systeem. Verder wordt verondersteld dat de krachtcomponenten $Q^a(y,x)$ afleidbaar zijn van een potentiële energiefunctie en dat het volledige systeem een cofactorsysteem is. De definitie van een cofactorsysteem, zowel op de Euclidische ruimte als op een Riemannvariëteit, herhalen we in Paragraaf 4.1. Merk op dat in de definitie van een cofactorsysteem op een Riemannvariëteit een speciale conforme Killingtensor opduikt. Een cofactorsysteem is immers een niet-conservatief systeem gegenereerd door een 1-vorm μ en een metrisch tensorveld g waarvoor er een niet-singuliere speciale conforme Killingtensor J bestaat zodat $D_J\mu = 0$.

In Paragraaf 4.2 herhalen we eerst een intrinsieke karakterisatie van gestuurde systemen om daarna de intrinsieke definitie van een gestuurd cofactorsysteem te beschouwen. Voor gestuurde cofactorsystemen gelden een aantal interessante resultaten, die we uitgebreid bespreken. Het is onder meer zo dat het systeem een tweede, weliswaar ontaarde, cofactorrepresentatie heeft. Het is met andere woorden een (ontaard) cofactorpaarsysteem. Dit geeft aanleiding tot een familie van $n+1$ kwadratische eerste integralen. Eén van deze is een eerste integraal van het sturend systeem, de andere n zijn, langs oplossingen van het sturend systeem, eerste integralen van het gestuurd systeem. De precieze oorsprong en constructie van deze tweede cofactorrepresentatie en de geassocieerde eerste integralen bespreken we in detail in Paragraaf 4.3.

Een nog opvallender resultaat is het volgende. Tijdens de bespreking van gestuurde cofactorsystemen op de Euclidische ruimte werd aangetoond dat er een tijdsafhankelijke canonische transformatie bestaat die ervoor zorgt dat de tijdsafhankelijkheid van het gestuurd systeem op een welbepaalde manier geëlimineerd wordt, zodat dit deel van het systeem geïdentificeerd kan worden met een Stäckelsysteem. De veralgemening van dit resultaat ligt niet voor de hand, het blijkt onmogelijk om dit resultaat te bekomen aan de hand van een tijdsafhankelijke punttransformatie. Onze theorie rond Forbats voorwaarden, ontwikkeld in Hoofdstuk 3, is bijgevolg niet bruikbaar. Daarom gaan we in de rest van Hoofdstuk 4 op zoek naar de oorsprong van deze canonische transformatie.

Voor de metriek van het gestuurd systeem identificeren we in Paragraaf 4.4 een speciale conforme Killingtensor $\bar{J}_2$. Tegen de verwachting in geeft $\bar{J}_2$ geen aanleiding tot een cofactorrepresentatie voor het gestuurd systeem, maar wel voor een aangepast gestuurd systeem. Maar deze speciale conforme Killingtensor speelt vooral een belangrijke rol in de bepaling van de geschikte canonische transformatie.

In Paragraaf 4.5 starten we met het introduceren van Darbouxcoördinaten voor de symplectische vorm geassocieerd aan de speciale conforme Killingtensor voor het volledige systeem. Deze Darbouxcoördinaten worden bekomen door een geschikte transformatie van de momenten. Maar gaandeweg komen we tot een nog betere keuze voor nieuwe momenten. Een transformatie van de momenten die ook rekening houdt met de gedeeltelijke ontkoppeling van het systeem en waarvoor we kunnen aantonen dat het een tijdsafhankelijke (standaard) canonische transformatie bepaalt voor het gestuurd systeem. We passen deze canonische transformatie toe in Paragraaf 4.6 en stellen vast dat de nieuwe Hamiltoniaan van het gestuurd systeem op een term lineair in de nieuwe momenten na, gelijk is aan een eerste integraal van het (gestuurd) systeem vermenigvuldigd met een functie enkel afhankelijk van de tijd (langs oplossingen van het sturend systeem). Deze eerste integraal kwamen we reeds tegen in de bespreking van de $n+1$ eerste integralen die volgen uit de dubbele cofactorrepresentatie van het volledig systeem.

Een tweede transformatie, ditmaal een (noodzakelijk) tijdsafhankelijke punttransformatie, waarbij de eigenfuncties van de speciale conforme Killingtensor $\bar{J}_2$ de nieuwe coördinaten bepalen, zorgt ervoor dat de term lineair in de momenten verdwijnt. De nieuwe Hamiltoniaan van het gestuurd systeem is dan gelijk aan het product van een eerste integraal van het gestuurd systeem en een functie van de tijd. Het is dan eenvoudig aan te tonen dat dit impliceert dat deze eerste integraal niet tijdsafhankelijk kan zijn en dat de tijdsafhankelijke Hamilton-Jacobivergelijking voor de Hamiltoniaan van het gestuurd systeem reduceert tot de autonome Hamilton-Jacobivergelijking voor deze eerste integraal. Bovendien blijkt deze eerste integraal te voldoen aan de voorwaarden voor een Stäckelsysteem.

Een aantal van de resultaten in Paragraaf 4.6 zijn gebaseerd op simpele, indirecte argumenten, maar worden ondersteund door ingewikkelde berekeningen in Paragraaf 4.7. We vervolledigen dan immers het recursief schema, reeds gestart in Paragraaf 4.3, om expliciete uitdrukkingen te bekomen voor alle eerste integralen. We kunnen dan uitdrukkelijk laten zien dat de eerste integralen van het gestuurd systeem langs oplossingen van het sturend systeem inderdaad tijdsonafhankelijk zijn.

We eindigen dit hoofdstuk met een aantal verduidelijkende voorbeelden in Paragraaf 4.8. Enerzijds lichten we toe hoe de ontaarde, gedeeltelijk ontkoppelde gevallen uit de studie van veralgemeende Hénon-Heiles systemen in onze theorie passen. Anderzijds willen we met een tweede, eenvoudig voorbeeld het begin van onze theorie illustreren, namelijk hoe vinden we coördinaten waarin het systeem gedeeltelijk ontkoppelt?

Referenties. Een groot deel van het werk voorgesteld in deze scriptie werd ondertussen reeds gepubliceerd. De resultaten uit Hoofdstuk 2 in verband met het liften van geometrische objecten naar $J^1\tau^*$ kunnen gevonden worden in [56] en de intrinsieke beschrijving van Forbats voorwaarden werd gepubliceerd in [61]. De nieuwe resulaten in het laatste hoofdstuk zijn gebaseerd op [55].

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